Recap of lecture 1:

Promote the band energy to the classical Hamiltonian:

$$H(\boldsymbol{r},\boldsymbol{k}) = E_{\boldsymbol{k}-e\boldsymbol{A}} + e\phi(\boldsymbol{r})$$

"Peierls substitution"

Write down the equations of motion:

$$\dot{\boldsymbol{r}} = \partial_{\boldsymbol{k}} E_{\boldsymbol{k}},$$

$$\dot{\boldsymbol{k}} = -\partial_{\boldsymbol{r}} E_{\boldsymbol{k}} + e\partial_{\boldsymbol{k}} E_{\boldsymbol{k}} \times \boldsymbol{B}.$$

Then perhaps solve the Boltzmann equation:

$$\partial_t f + \dot{\boldsymbol{r}} \boldsymbol{\nabla} f + \dot{\boldsymbol{k}} \partial_{\boldsymbol{k}} f = \hat{I}_{st}$$

We would like to describe deviations from this picture, i.e. the departure from the classical point of view.

Position operator in the band representation

Wave function in the band representation:

$$|\psi\rangle = \sum_{n\boldsymbol{k}} c_{n\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} |u_{n\boldsymbol{k}}\rangle, \ \ \hat{\boldsymbol{r}} c_{n\boldsymbol{k}} = ?$$

Minimal derivation:

$$\hat{\boldsymbol{r}}|\psi\rangle = \sum_{n\boldsymbol{k}} c_{n\boldsymbol{k}} \frac{1}{i} (\partial_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}}) |u_{n\boldsymbol{k}}\rangle = \sum_{n\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} i\partial_{\boldsymbol{k}} (c_{n\boldsymbol{k}} |u_{n\boldsymbol{k}}\rangle)$$

$$= \sum_{n\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} (i\partial_{\boldsymbol{k}} c_{n\boldsymbol{k}} |u_{n\boldsymbol{k}} + c_{n\boldsymbol{k}} i\partial_{\boldsymbol{k}} |u_{n\boldsymbol{k}}\rangle)$$

$$\to \sum_{n\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} |u_{n\boldsymbol{k}}\rangle (i\partial_{\boldsymbol{k}} + i\langle u_{n\boldsymbol{k}} | \partial_{\boldsymbol{k}} | u_{n\boldsymbol{k}}\rangle) c_{n\boldsymbol{k}} \equiv \sum_{n\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} |u_{n\boldsymbol{k}}\rangle (\hat{\boldsymbol{r}} c_{n\boldsymbol{k}})$$

Position operator projected onto a band in a crystal:

$$\hat{\mathbf{r}} = i\nabla_{\mathbf{k}} + \mathbf{A}_{\mathbf{k}}, \ \ \mathbf{A}_{\mathbf{k}} = i\langle u_{n\mathbf{k}} | \partial_{\mathbf{k}} u_{n\mathbf{k}} \rangle$$

Lattice coordinate

Coordinate within the unit cell

Semiclassical motional in external fields

Proceed by comparison:

$$\hat{\mathbf{p}} = \frac{1}{i}\nabla_{\mathbf{r}} - e\boldsymbol{A}_{\mathbf{r}}$$

$\hat{\mathbf{r}} = i\nabla_{\mathbf{p}} + \mathbf{A}_{\mathbf{p}}$ - looks like a vector potential in momentum space

Motion in ***external*** fields is semiclassical:

$$\begin{aligned}
\dot{\mathbf{p}} &= -e\frac{\partial\phi}{\partial\mathbf{r}} + e\dot{\mathbf{r}} \times \mathbf{B}, \;\; \mathbf{B} = \nabla_{\mathbf{r}} \times \boldsymbol{A}_{\mathbf{r}} \\
\dot{\mathbf{r}} &= \frac{\partial\epsilon_{n\mathbf{p}}}{\partial\mathbf{p}} - \dot{\mathbf{p}} \times \boldsymbol{\Omega}_{n\mathbf{p}}, \;\; \boldsymbol{\Omega}_{n\mathbf{p}} = \nabla_{\mathbf{p}} \times \mathbf{A}_{\mathbf{p}}
\end{aligned}$$

$$\boldsymbol{\Omega}_{n\mathbf{p}} = i\langle\partial_{\mathbf{p}}u_{n\mathbf{p}}| \times |\partial_{\mathbf{p}}u_{n\mathbf{p}}\rangle$$

Application: anomalous Hall effect (B=0)

$$\begin{aligned} \dot{\mathbf{p}} &= e\boldsymbol{E} + e\dot{\mathbf{r}} \times \mathbf{B} \\ \dot{\mathbf{r}} &= \frac{\partial \epsilon_{n\mathbf{p}}}{\partial \mathbf{p}} - \dot{\mathbf{p}} \times \boldsymbol{\Omega}_{n\mathbf{p}} \end{aligned} \quad \xrightarrow{B=0} \quad \begin{aligned} \dot{\mathbf{p}} &= e\mathbf{E} \\ \dot{\mathbf{r}} &= \frac{\partial \epsilon_{n\mathbf{p}}}{\partial \mathbf{p}} - e\mathbf{E} \times \boldsymbol{\Omega}_{n\mathbf{p}} \end{aligned}$$

$$\boldsymbol{j}^{\mathrm{AHE}} = e^2 \int_{\mathbf{p}} \boldsymbol{\Omega}_{n\mathbf{p}} \times \boldsymbol{E} f_{n\mathbf{p}}, \;\; \sigma_{ab}^{\mathrm{AHE}} = -e^2 \epsilon_{abc} \int_{\mathbf{p}} \boldsymbol{\Omega}^c_{n\mathbf{p}} f_{n\mathbf{p}}.$$

<u>Historical time scales:</u>
discovery - E. Hall in 1881,
relation to the spin-orbit coupling - 1954 by Karplus&Luttinger,
relation to band geometry - 1982 by TKNN

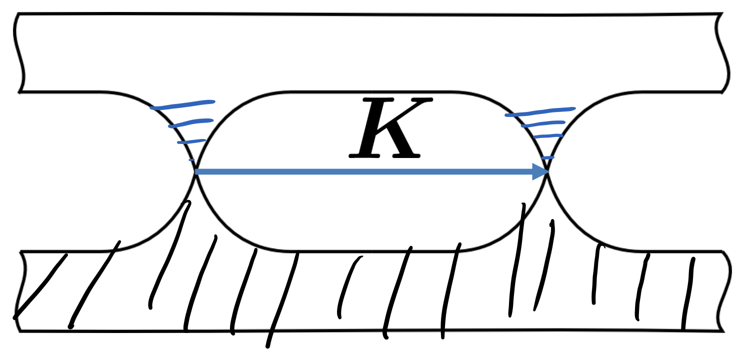
Compare to superconductivity&BCS: 1911-1957

Application: anomalous Hall effect (B=0)

$$\begin{aligned}\dot{\mathbf{p}} &= e\boldsymbol{E} + e\dot{\mathbf{r}} \times \mathbf{B} \\ \dot{\mathbf{r}} &= \frac{\partial \epsilon_{n\mathbf{p}}}{\partial \mathbf{p}} - \dot{\mathbf{p}} \times \boldsymbol{\Omega}_{n\mathbf{p}}\end{aligned} \quad \xrightarrow{B=0} \quad \begin{aligned}\dot{\mathbf{p}} &= e\mathbf{E} \\ \dot{\mathbf{r}} &= \frac{\partial \epsilon_{n\mathbf{p}}}{\partial \mathbf{p}} - e\mathbf{E} \times \boldsymbol{\Omega}_{n\mathbf{p}}\end{aligned}$$

$$\boldsymbol{j}^{\mathrm{AHE}} = e^2 \int_{\mathbf{p}} \boldsymbol{\Omega}_{n\mathbf{p}} \times \boldsymbol{E} f_{n\mathbf{p}}, \;\; \sigma_{ab}^{\mathrm{AHE}} = -e^2 \epsilon_{abc} \int_{\mathbf{p}} \boldsymbol{\Omega}_{n\mathbf{p}}^{c} f_{n\mathbf{p}}.$$

For a Weyl semimetal:



$$\int_{\boldsymbol{k}} \Omega_{\boldsymbol{k}}^{a} = \int_{\boldsymbol{k}} \Omega_{\boldsymbol{k}}^{b} \delta_{ab} = \int_{\boldsymbol{k}} \Omega_{\boldsymbol{k}}^{b} \partial_{k_b} k_a \stackrel{?}{=} -\int_{\boldsymbol{k}} k_a (\partial_{\boldsymbol{k}} \cdot \boldsymbol{\Omega}_{\boldsymbol{k}})$$

$$\sigma_{ab}^{\mathrm{AHE}} = \frac{e^2}{2\pi h} \epsilon_{abc} K_c$$

"almost quantized" 3D Hall effect.
K is defined up to a reciprocal lattice vector. (Haldane, PRL 2004)

Motion in magnetic field

$$
\begin{aligned}
\dot{\mathbf{p}} &= e\boldsymbol{E} + e\dot{\mathbf{r}} \times \mathbf{B} \\
\dot{\mathbf{r}} &= \mathbf{v}_{\mathbf{p}} - \dot{\mathbf{p}} \times \boldsymbol{\Omega}_{n\mathbf{p}}
\end{aligned}
$$

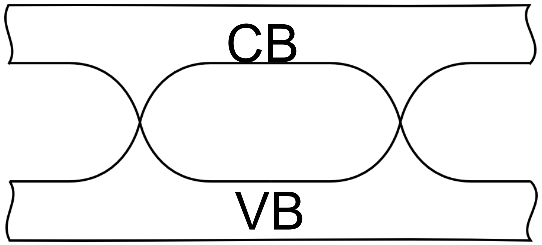
Chiral anomaly

$$
\begin{aligned}
\dot{\mathbf{p}} &= \frac{1}{D_{\boldsymbol{B}}}\left(e\boldsymbol{E} + e\mathbf{v}_{\mathbf{p}} \times \mathbf{B} - e^2(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_{\mathbf{p}}\right) \\
\dot{\mathbf{r}} &= \frac{1}{D_{\boldsymbol{B}}}\left(\mathbf{v}_{\mathbf{p}} - e\boldsymbol{E} \times \boldsymbol{\Omega}_{n\mathbf{p}} - e(\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\Omega}_{\mathbf{p}})\boldsymbol{B}\right)
\end{aligned}
$$

AHE Chiral magnetic effect

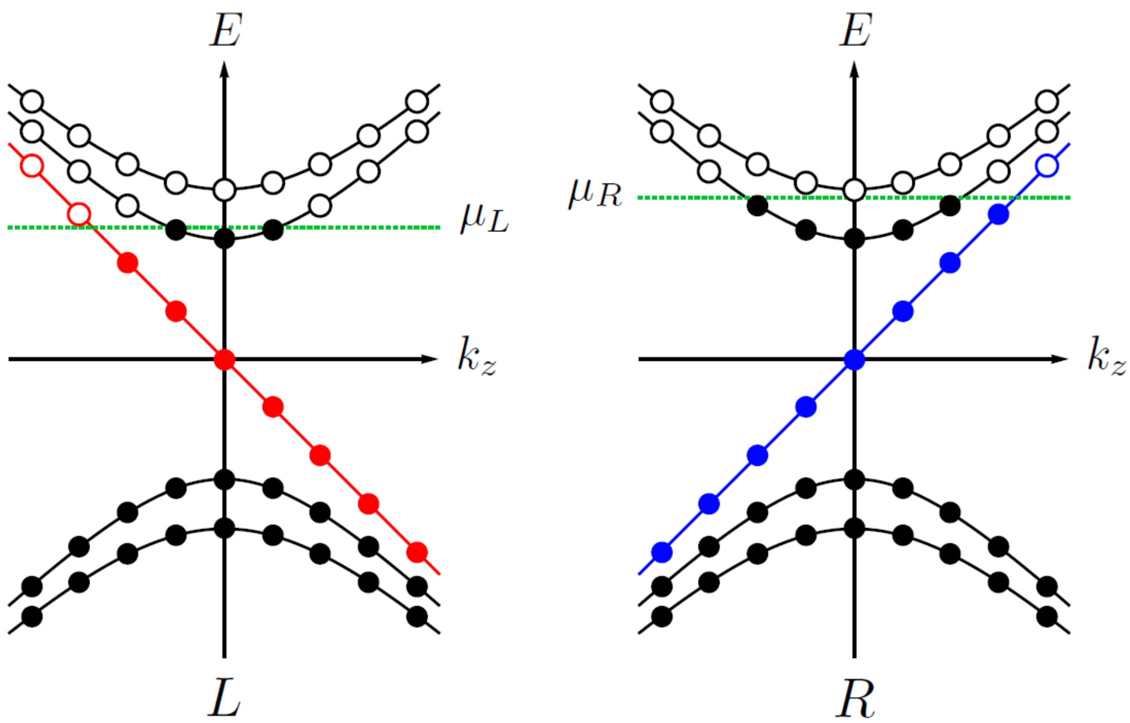
$$
D_{\boldsymbol{B}} = 1 - e\boldsymbol{B}\boldsymbol{\Omega}_{\boldsymbol{p}}
$$

LL interpretation: chiral anomaly



$$\Rightarrow$$

$$\vec{B} = (0, 0, B),$$
$$H = \pm v \left[\underbrace{\vec{\sigma}_\perp (\vec{p}_\perp - e\vec{A})}_{\text{“graphene”}} + \underbrace{\sigma_z p_z}_{\text{“gap”}} \right]$$

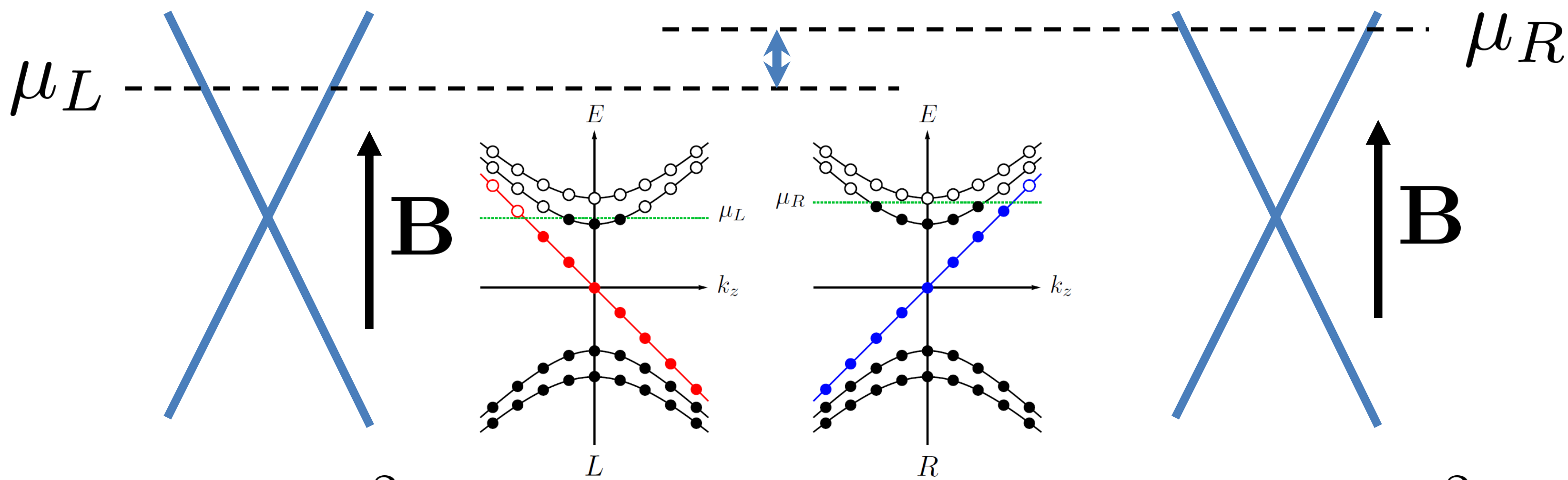


$$\boxed{\dot{N}_R - \dot{N}_L = \frac{e^2}{2\pi^2\hbar^2 c}\mathbf{E}\cdot\mathbf{B}}$$

“3D chiral anomaly”

(S. L. Adler, 1969 ; J. S. Bell and R. Jackiw, 1969; Nielsen&Ninomiya, 1983)

LL interpretation: CME



$$\mathbf{j}_L^{CME} = \frac{e^2\mu_L}{4\pi^2}\mathbf{B}$$

(Vilenkin, 1980)

$$\mathbf{j}_R^{CME} = -\frac{e^2\mu_R}{4\pi^2}\mathbf{B}$$

$$\mathbf{j}_{\omega=0}^{CME} = \frac{e^2(\mu_L-\mu_R)}{4\pi^2}\mathbf{B}$$

$$\mathbf{j}_{\omega\neq 0}^{CME} = \frac{e^2(\mu_L-\mu_R)}{12\pi^2}\mathbf{B}$$

(Kharzeev, Warringa, 2009;
Son, Yamamoto, 2013)

Chiral magnetic effect

$$\boldsymbol{j} = e \int_{\boldsymbol{p}} D_{\boldsymbol{B}} \dot{\boldsymbol{r}}$$

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{1}{D_{\boldsymbol{B}}}\left(e\boldsymbol{E} + e\mathbf{v}_{\mathbf{p}} \times \mathbf{B} - e^2(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_{\mathbf{p}}\right) \\ \dot{\mathbf{r}} &= \frac{1}{D_{\boldsymbol{B}}}\left(\mathbf{v}_{\mathbf{p}} - e\boldsymbol{E} \times \boldsymbol{\Omega}_{n\mathbf{p}} - e(\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\Omega}_{\mathbf{p}})\boldsymbol{B}\right) \end{aligned}$$

$$\boldsymbol{j}_{CME} = \left[-e^2 \sum_n \int_{\boldsymbol{p}} (\boldsymbol{v}_{n\boldsymbol{p}} \cdot \boldsymbol{\Omega}_{n\boldsymbol{p}}) f_{n\boldsymbol{p}}\right] \boldsymbol{B}$$

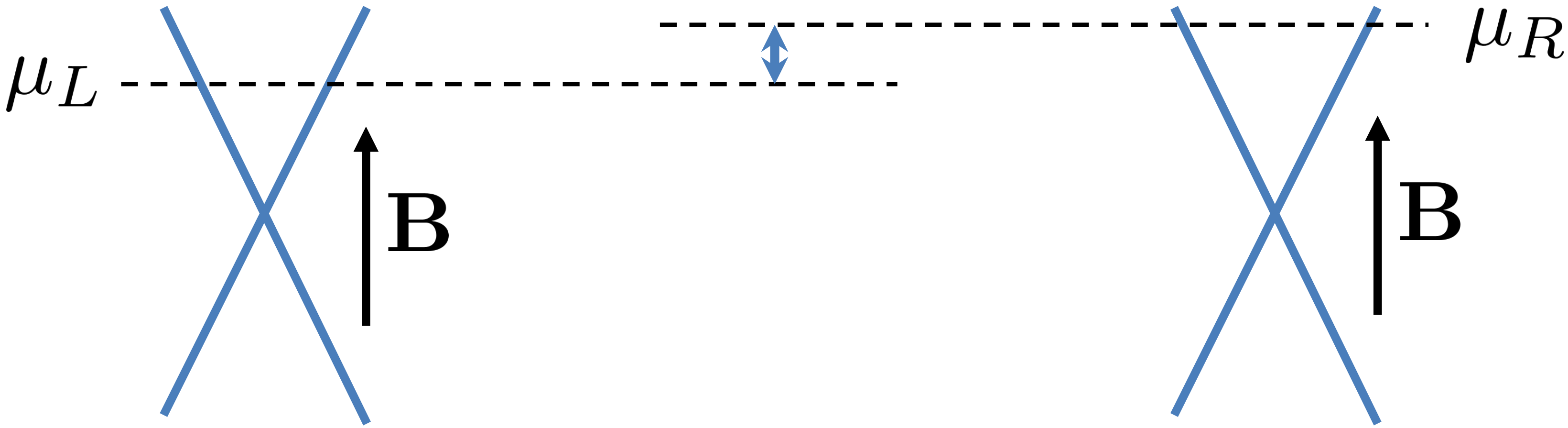
Looks like a "Fermi sea" current

However, using $\boldsymbol{v}_{n\boldsymbol{p}} = \partial_{\boldsymbol{p}} \epsilon_{n\boldsymbol{p}}$ and integrating by parts one arrives at

$$\begin{aligned} \boldsymbol{j}_{CME} &= \left[e^2 \sum_n \int_{\boldsymbol{p}} (\epsilon_{n\boldsymbol{p}} \boldsymbol{\nabla}_{\boldsymbol{p}} \cdot \boldsymbol{\Omega}_{n\boldsymbol{p}}) f_{n\boldsymbol{p}} + \epsilon_{n\boldsymbol{p}} \boldsymbol{\Omega}_{n\boldsymbol{p}} \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} f_{n\boldsymbol{p}}\right] \boldsymbol{B} \\ &= \left[\frac{e^2}{4\pi^2} \sum_W \mu_W Q_W\right] \boldsymbol{B} \end{aligned}$$

Berry monopoles are required for *static* CME

CME in a Weyl semimetal



$$\mathbf{j}_L^{CME} = \frac{e^2 \mu_L}{4\pi^2}\mathbf{B}$$

(Vilenkin, 1980)

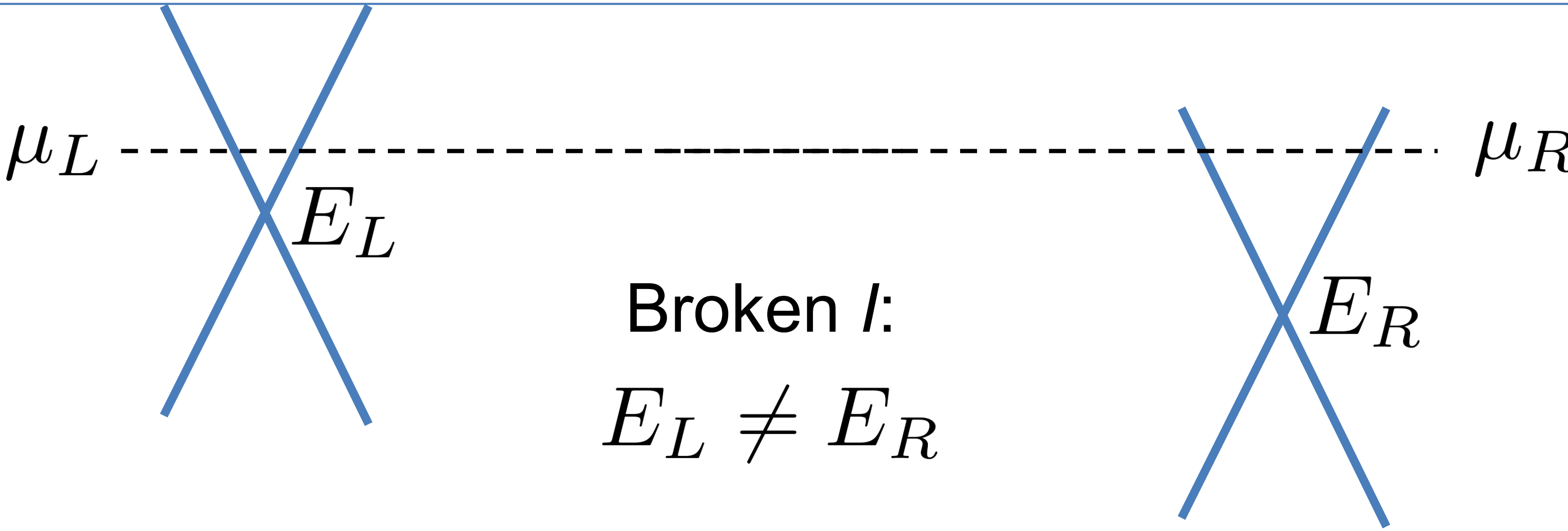
$$\mathbf{j}_R^{CME} = -\frac{e^2 \mu_R}{4\pi^2}\mathbf{B}$$

$$\mathbf{j}_{\omega=0}^{CME} = \frac{e^2(\mu_L - \mu_R)}{4\pi^2}\mathbf{B}$$

$$\mathbf{j}_{\omega\neq 0}^{CME} = \frac{e^2(\mu_L - \mu_R)}{12\pi^2}\mathbf{B}$$

(Kharzeev, Warringa, 2009;
Son, Yamamoto, 2013)

There is only dynamic CME in equilibrium crystals



Broken *I*:

$$E_L \neq E_R$$

$$\mathbf{j}^{CME}_{\omega=0} = 0$$

(Zhou, Jiang, Niu, Shi, Chin. Phys. Lett., 2013; Vazifeh, Franz, PRL, 2013)

Physical reason: $\boldsymbol{j} \propto \boldsymbol{B}$ implies $\boldsymbol{M} \propto \boldsymbol{A}$ in equilibrium

(Levitov, Nazarov, Eliashberg, JETP 1985)

For $\mathbf{j}^{CME}_{\omega\neq 0}$ see Chen, Wu, Burkov, PRB, 2013
Chang, Yang PRB 2015;
Ma, Pesin, PRB 2015;
Zhong, Moore, Souza, PRL 2016

The chiral anomaly

$$\partial_t f_{\boldsymbol{p}} + \dot{\boldsymbol{p}} \partial_{\boldsymbol{p}} f_{eq} = \hat{I}_{st}^{intra}$$

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{1}{D_{\boldsymbol{B}}} \left(e\boldsymbol{E} + e\mathbf{v}_{\mathbf{p}} \times \mathbf{B} - e^2 (\boldsymbol{E} \cdot \boldsymbol{B}) \boldsymbol{\Omega}_{\mathbf{p}} \right) \\ \dot{\mathbf{r}} &= \frac{1}{D_{\boldsymbol{B}}} \left(\mathbf{v}_{\mathbf{p}} - e\boldsymbol{E} \times \boldsymbol{\Omega}_{n\mathbf{p}} - e(\mathbf{v}_{\mathbf{p}} \cdot \boldsymbol{\Omega}_{\mathbf{p}}) \boldsymbol{B} \right) \end{aligned}$$

Equation for the density in a given valley: $\rho_W = e \int_{\boldsymbol{p}} D_{\boldsymbol{B}} f_{\boldsymbol{p}}$

$$\partial_t \rho_W = e^3 (\boldsymbol{E} \cdot \boldsymbol{B}) \int_{\boldsymbol{p}} \boldsymbol{\Omega}_{\boldsymbol{p}} \partial_{\boldsymbol{p}} f_{eq} = e^3 (\boldsymbol{E} \cdot \boldsymbol{B}) \int_{\boldsymbol{p}} (\boldsymbol{\Omega}_{\boldsymbol{p}} \cdot \boldsymbol{v}_{\boldsymbol{p}}) \partial_{\varepsilon_{\boldsymbol{p}}} f_{eq}$$

$$= \frac{e^3}{4\pi^2} Q_W \boldsymbol{E} \cdot \boldsymbol{B}$$

Total charge near an individual Weyl point is not conserved

The net charge conservation is ensured by “Berry neutrality”:

$$\sum_W Q_W = 0$$

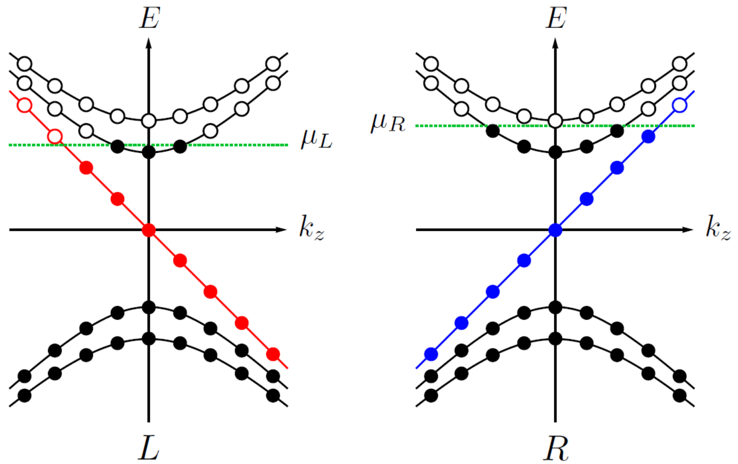
"Anomalous" transport theory in WS

(for a hydrodynamic description see Lucas, Richardson, Sachdev, PNAS 2016)

The currents include the chiral modes contributions:

$$\mathbf{j}^{R,L} = -\frac{\sigma}{e}\boldsymbol{\nabla}\mu_{\text{ec}}^{R,L} \pm \underline{\underline{\frac{e^2\mathbf{B}}{4\pi^2\hbar^2 c}\mu^{R,L}}}.$$

$$\mu_{\text{ec}}^{R,L} = \mu^{R,L} + e\phi$$



The continuity equations include the anomalous divergences:

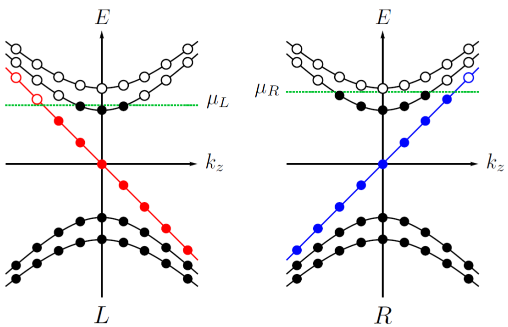
$$\boldsymbol{\nabla}\cdot\mathbf{j}^{R,L} + \partial_t\rho^{R,L} = \pm\frac{e^3}{4\pi^2\hbar^2 c}\mathbf{E}\cdot\mathbf{B}$$

The final stationary transport equations contain only $\mu_{\text{ec}}^{R,L}$

$$\boxed{-\frac{\sigma}{e}\nabla^2\mu_{\text{ec}}^{R,L} \pm \frac{e^2}{h^2}\boldsymbol{B}\cdot\boldsymbol{\nabla}\mu_{\text{ec}}^{R,L} = \mp\frac{e\nu_{3\text{D}}}{2\tau_v}(\mu_{\text{ec}}^{R} - \mu_{\text{ec}}^{L})}$$

Negative magnetoresistance from the chiral anomaly

(Son, Spivak, PRB 2012)



For clarity: $-\boldsymbol{\nabla}\mu_{ec} \to e\boldsymbol{E}$

Use chiral anomaly to generate imbalance:

$$\frac{e^3}{4\pi^2}B_z E_z = \frac{e\nu_{3\mathrm{D}}}{2\tau_v}(\mu^R - \mu^L) \Rightarrow \mu^R - \mu^L = \frac{e^2}{2\pi^2}\frac{\tau_v}{\nu_{3D}}B_z E_z$$

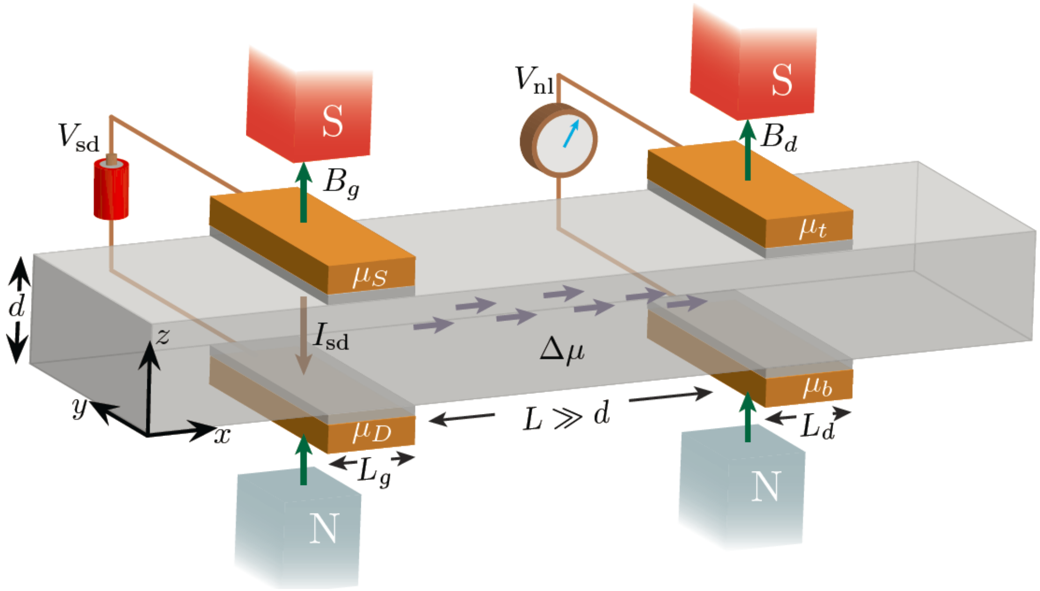
Convert the imbalance into “more conductivity” by the CME:

$$\delta j = \frac{e^2}{4\pi^2}(\mu^R - \mu^L) \Rightarrow \delta\sigma_{zz} = \frac{e^4}{8\pi^4}\frac{\tau_v}{\nu_{3D}}B_z^2$$

$$\frac{\delta\sigma_{zz}}{|\sigma_z z(B) - \sigma_{zz}(0)|} \sim \frac{\tau_v}{\tau}\frac{1}{\mu^2\tau^2}$$ can be large

For a discussion of experimental issues, see Liang et al., PRX 2018

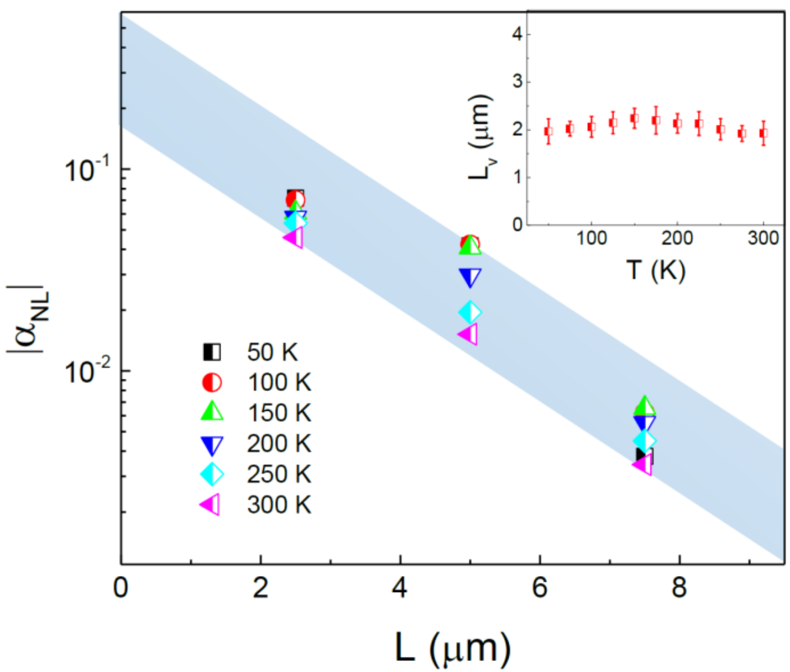
Non-local transport from chiral anomaly/CME



$$\frac{|V_{\rm nl}(x)|}{V_{\rm SD}} \propto e^{-x/\ell_v}, \quad \ell_v = \sqrt{D\tau_v} \gg d$$

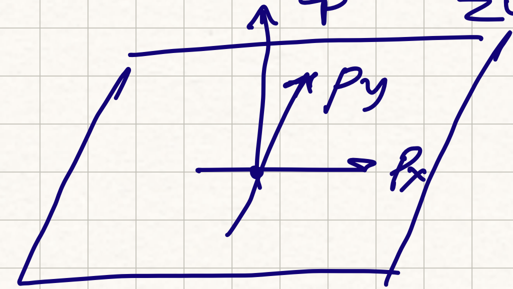
S. Parameswaran, T. Grover, D. Abanin, **DP,** A. Vishwanath, PRX 4, 031035 (2014)

Measurement:
C. Zhang, et al
Nature, 2017



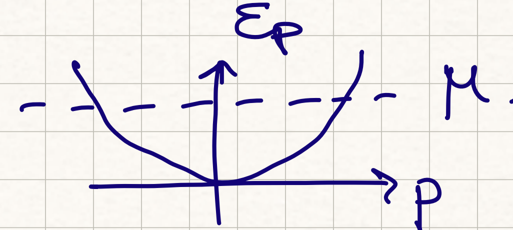
Intrinsic and extrinsic AHE in a simple model

$$\varepsilon_p = \frac{p_x^2+p_y^2}{2m}$$



2D metal with isotropic dispersion, and constant $\vec{\Omega} = (0,0,\Omega)$.

Intrinsic Hall conductivity:



$$\vec{j}^{\,int} = e\int_p \vec{\Omega}\times e\vec{E}\cdot f_{eq},$$

$$\sigma^{int}_{ab} = e^2\cdot\underbrace{\int\frac{d^2p}{(2\pi\hbar)^2}\cdot f_{eq}}_{n}\cdot\left[-\varepsilon_{abc}\Omega_c\right]$$

$$= -\varepsilon_{abc}\Omega_c\cdot e^2 n = -\varepsilon_{ab}(ne^2\Omega)$$

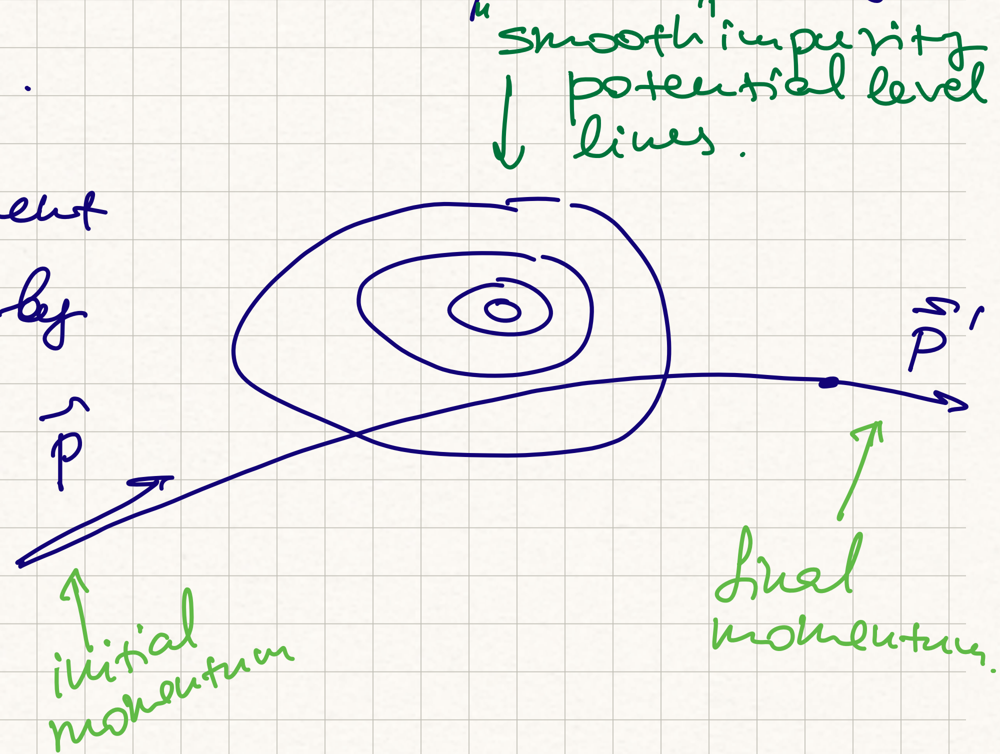
$$\boxed{\sigma^{int}_{ab} = -\varepsilon_{ab}\, ne^2\Omega}$$

Extrinsic contributions to the AHE

It turns out that beside the intrinsic – Berry curvature – part of the anomalous Hall conductivity, there are ones related to impurity scattering. These are known as "side jump" and "skew scattering" contributions.

Side jump: shift of a wave packet center upon scattering from an impurity.

Qualitative picture: displacement due to anomalous velocity caused by the impurity potential (weak and smooth)



$$\vec{v}_{an} = \vec{\Omega} \times e\vec{E}_{imp}, \qquad e\vec{E}_{imp} = \frac{d\vec{p}}{dt}$$

$$\delta \vec{r}_{p'p} = \int_{t_i}^{t_f} dt\, \vec{v}_{an} = \vec{\Omega} \times (\vec{p}\,' - \vec{p}) \qquad \Omega \simeq \mathrm{const}(\vec{p})$$

the exact expression for spherically symmetric potential:

$$\delta\vec{r}_{p'p} = i\langle u_{p'} | \partial_{p'} u_{p'}\rangle - i\langle u_p | \partial_p u_p\rangle - (\partial_p + \partial_{p'})\operatorname{Arg}\langle u_{p'} | u_p\rangle,$$

which gives for small-angle scattering:

$$\delta\vec{r}^{\,\text{sm. angle}}_{pp'} \simeq \vec{\Omega}\left(\frac{p+p'}{2}\right)\times(\vec{p}\,' - \vec{p}) .$$

→ similar to the expr. from the "qualitative picture")

SJ contribution to AHE

Side jumps have two effects on transport:

- repeated shifts add up to mimic a change in the velocity
- shift of a particle during a collision changes its potential energy in the external field, which must be taken into account in energy conservation.

Side-jump accumulation velocity:

$$\vec{v}^{\,sj}_{\vec{p}} = \int_{p'} w_{pp'}\, \delta r_{p'p}\, \delta(\xi_p - \xi_{p'}) \sim \text{“}\frac{\Delta r}{\Delta t}\text{”} \sim$$ average shift when $\vec{p} \to$ all other $\vec{p}\,'$.

↳ order of indices is important!

Contribution to the electric current:

$$\vec{j}^{\,sj} = \int_p \vec{v}_p^{\,sj} \cdot \delta f_p$$ (f_{eq} gives zero since $\delta r_{pp'} = -\delta r_{p'p}$)

Simplification: $w_{pp'} \to w_0$, then for small angle scattering

$$\vec{v}^{\,sj}_{\vec{p}} = \int_{p'} w_0 \cdot \vec{\Omega} \times (\vec{p}\,' - \vec{p}\,)\, \delta(\xi_p - \xi_{p'}) = -\vec{\Omega} \times \vec{p} \cdot \int_{p'} w_0\, \delta(\xi_p - \xi_{p'}).$$

But $\frac{1}{\tau_{tr}} = \int_{p'} w_0 (1 - \cancel{\hat{e}_p \hat{e}_{p'}})\, \delta(\xi_p - \xi_{p'})$, so

$$\vec{v}^{\,sj}_{\vec{p}} = -\frac{\vec{\Omega} \times \vec{p}}{\tau_{tr}}$$

We can now calculate the side-jump-accumulation contribution to the current:

$$\vec{j}^{\,sja} = e \cdot \int_p \cdot \vec{v}_{\vec{p}}^{\,sja} \cdot \delta f_{\vec{p}} \quad \Big\{ \quad \delta f_{\vec{p}} = -e\vec{E} \cdot \partial_p f_{eq} \cdot \tau_{tr} \ - \text{obtained this before.}$$

$$= -e \int_p \cdot \frac{\vec{\Omega} \times \vec{p}}{\cancel{\tau_{tr}}} \cdot \cancel{\tau_{tr}} \left(-e\vec{E} \cdot \partial_{\vec{p}} f_{eq} \right)$$

τ_{tr} got cancelled! The result looks "intrinsic".

We can extract the contribution to conductivity: $\vec{p} \to m\vec{v}$

$$\sigma_{ab}^{sja} = +e^2 \int_p \cdot \varepsilon_{acl} \cdot \Omega_c \, p_l \cdot \frac{\partial f_{eq}}{\partial p_b} \overset{\text{by parts}}{=} -e^2 \cdot \int_p \varepsilon_{acl} \Omega_c \cdot f_{eq} \cdot \delta_{lb}$$

$$= -e^2 \cdot \int_p \varepsilon_{acb} \, \Omega_c \ f_{eq} = + e^2 \cdot n \Omega \, \varepsilon_{ab}$$

compare to the "true" intrinsic conductivity sign (opposite)

Modification of the energy-conserving δ-function

Shift of $\delta r_{pp'}$ upon scattering ($\vec{p}' \to \vec{p}$) leads to the external field doing work on the carriers: $\mathcal{U} = e\vec{E}\cdot\delta r_{pp'}$. It needs to be added to the energy conservation equation:

$$\varepsilon_{p'} + e\vec{E}\delta\vec{r}_{pp'} = \varepsilon_p \Rightarrow \delta(\varepsilon_p - \varepsilon_{p'}) \to \delta(\varepsilon_p - \varepsilon_{p'} - e\vec{E}\,\delta\vec{r}_{pp'})$$

For $\vec{p} \to \vec{p}'$ we get

$$\varepsilon_p + e\vec{E}\delta r_{p'p} = \varepsilon_{p'} \Rightarrow \delta(\varepsilon_p - \varepsilon_{p'}) \to \delta(\varepsilon_p - \varepsilon_{p'} + e\vec{E}\,\delta\vec{r}_{p'p})$$

$$= \delta(\varepsilon_p - \varepsilon_{p'} - e\vec{E}\,\delta\vec{r}_{pp'}), \text{ as before.}$$

This modification makes the collision integral not vanish when evaluated for $f_{p,p'} = f_{eq}(\varepsilon_{p,p'})$:

$$I = -\sum_{p'} \left(W_{p'p} f_p - W_{pp'} f_{p'}\right) \delta(\varepsilon_p - \varepsilon_{p'} - e\vec{E}\,\delta\vec{r}_{pp'}) \equiv I_0 + \delta I_E =$$

$$= I_0 \underbrace{- \sum_{p'} \left[W_{p'p} f_p - W_{pp'} f_{p'}\right] \frac{\partial}{\partial \varepsilon_p} \delta(\varepsilon_p - \varepsilon_{p'}) \cdot \left(-e\vec{E}\,\delta\vec{r}_{pp'}\right)}_{\delta I_E\text{, a new generation term in the kinetic equation.}}$$

Evaluate δI_E with the same assumptions as before:

$$\delta I_E = + \int_{p'} W_0 \underbrace{\left(f_p - f_{p'}\right)}_{\text{can set these to } f_{eq}\text{, since the term is already } O(\vec{E})} \cdot \partial_{\xi_p} \delta(\xi_p - \xi_{p'})\, e\vec{E}\left(\vec{\Omega}\times(\vec{p} - \cancel{\vec{p}'})\right)$$

$$= e\vec{E}\cdot\vec{\Omega}\times\vec{p}\left[\cancel{f_p \partial_{\xi_p} \cdot \underbrace{\int_{p'} W_0\, \delta(\xi_p - \xi_{p'})}_{W_0 \cdot \nu(\xi_p) = \text{const}(p) \text{ for } \xi_p = \frac{p^2}{2m} \text{ in 2D}}}^{\to 0} - \partial_{\xi_p}\cdot \int_{p'} W_0 f_{p'}\, \delta(\xi_p - \xi_{p'})\right]$$

$$= e\vec{E}\cdot\vec{\Omega}\times\vec{p}\cdot\left[-\partial_{\xi_p}\left(f_{eq}\cdot\frac{1}{\tau_{tr}}\right)\right] = -\frac{1}{\tau_{tr}}\left(e\vec{E}\cdot\vec{\Omega}\times\vec{p}\right)\cdot\partial_{\xi_p} f_{eq}$$

The correction to the distribution function due to δI_E comes from balancing it against I_0:

$$-\frac{\delta f^{ad}}{\tau_{tr}} - \frac{1}{\tau_{tr}}\, e\vec{E}\left(\vec{\Omega}\times\vec{p}\right)\partial_{\xi_p} f_{eq} = 0 \Rightarrow \delta f^{ad} = -e\vec{E}\left(\vec{\Omega}\times\vec{p}\right)\partial_{\xi_p} f_{eq}.$$

"ad" → "anomalous distribution".

The corresponding current is

$$\vec{j}^{\,ad} = e\int_p \vec{v}_p \cdot \delta f^{ad} = -e^2 \int_p \left(e\vec{E}\cdot\vec{\Omega}\times\vec{p}\right)\cdot \underbrace{\frac{\partial f_{eq}}{\partial \vec{p}}}_{=\vec{v}_p \cdot \frac{\partial f_{eq}}{\partial \xi_p}}.$$

The corresponding conductivity is

$$\sigma^{ad}_{ab} = -e^2 \int_p \varepsilon_{bcl}\, \Omega_c\, p_e \frac{\partial f_{eq}}{\partial p_a} = +e^2 \varepsilon_{bca}\, \Omega_c \int_p f_{eq} =$$

different a's!

$$= + \varepsilon_{ab} \cdot n e^2 \Omega$$, just like for side jump accumulation.

The overall result for side jump mechanism:

$$\sigma^{int+si}_{ab} = \sigma^{\Omega}_{ab} + \sigma^{sja}_{ab} + \sigma^{ad}_{ab} = -\sigma^{\Omega}_{ab}$$; the effect of side jumps is to "flip" the sign of Berry curvature!

Onsager relations for the conductivity tensor

Def. of the conductivity tensor:

$t \to t' \leq t$: causality

$$j_a(\vec{r},t) = \int d^3r' \int_{-\infty}^{t} dt' \, \sigma_{ab}(\vec{r}-\vec{r}', t-t') \, E_b(\vec{r}', t').$$

In Fourier space:

$$j_a(\vec{r},t) = \int \frac{d^3q}{(2\pi)^3} \cdot \frac{d\omega}{2\pi} \, e^{i\vec{q}\vec{r} - i\omega t} \, j_a(\vec{q},\omega),$$ and the same for E_a.

Then

$$\boxed{j_a(\vec{q},\omega) = \sigma_{ab}(\vec{q},\omega) \, E_b(\vec{q},\omega)}$$

Because of causality, $\sigma_{ab}(\omega)$ is an analytic function of ω in the upper half-plane of complex frequencies.

In the presence of a $\vec{B}$-field or magnetization, we have $\sigma(\vec{q},\omega,\vec{B})$.

It turns out that microscopic time-reversibility of laws of physics imposes certain constraints on $\sigma_{ab}(\vec{q}, \omega, \vec{B})$. The following discussion is informal, but allows for easy remembering.

Tensor σ_{ab} contains dissipative and reactive parts:

dissipative part - absorption of energy,

reactive part - refractive index of the medium.

Which is which then? Assume monochromatic $E \propto \cos\omega t$. The average absorbed power is $\propto \underset{\text{over osc. period}}{\langle \vec{j} \cdot \vec{E} \rangle}$ (Joule heat), or

$$Q = \frac{1}{2} \operatorname{Re} E_a^* j_a = \frac{1}{2} \operatorname{Re} E_a^* \sigma_{ab} E_b = \frac{1}{4} \Big(E_a^* \sigma_{ab} E_b +$$

$$+ \underset{a \leftrightarrow b}{E_a \sigma_{ab}^* E_b^*} \Big) = \frac{1}{2} E_a^* \underbrace{\left(\frac{\sigma_{ab} + \sigma_{ba}^*}{2} \right)}_{} E_b.$$

σ_{ab}^H - Hermitian part of σ, $(\sigma_{ba}^H)^* = \sigma_{ab}^H$

One can also show that the antihermitian part determines the dispersion of EM waves.

Basic physical considerations:

- dissipation should be even w.r.t. $t \to -t$
- the reactive part is odd under $t \to -t$

Reality conditions: since $j^*(\vec{r},t) = j(\vec{r},t)$, and $E^*(r,t) = E(r,t)$, one can easily show that $\sigma^*_{ab}(\vec{q},\omega,\vec{B}) = \sigma_{ab}(-\vec{q},-\omega,\vec{B})$.

Combine time reversal, $\omega \to -\omega$, $\vec{B} \to -\vec{B}$, with reality conditions:

$$\sigma^H_{ab}(\vec{q},\omega,\vec{B}) = \sigma^H_{ab}(\vec{q},-\omega,-\vec{B})$$

$$\sigma^A_{ab}(\vec{q},\omega,\vec{B}) = -\sigma^A_{ab}(\vec{q},-\omega,-\vec{B}) \quad , \text{and}$$

$$\sigma^*_{ab}(\vec{q},\omega,\vec{B}) = \sigma_{ab}(-\vec{q},-\omega,\vec{B})$$

$$\sigma^H_{ab} = \left(\sigma^H_{ba}\right)^*$$

$$\sigma^A_{ab} = -\left(\sigma^A_{ba}\right)^*$$

to obtain

$$\left[\sigma_{ab}(\vec{q},\omega,\vec{B})\right]^* = \left[\sigma^{H} + \sigma^{A}\right]^* = \left[\sigma^{H}_{ab}(\vec{q},-\omega,-\vec{B}) - \sigma^{A}_{ab}(\vec{q},-\omega,-\vec{B})\right]^*$$

$$= \sigma^{H}_{ba}(\vec{q},-\omega,-\vec{B}) + \sigma^{A}_{ba}(\vec{q},-\omega,-\vec{B}) = \sigma_{ba}(\vec{q},-\omega,-\vec{B})$$

$= \sigma_{ab}(-\vec{q},-\omega,\vec{B})$, and conclude that

$\sigma_{ab}(-\vec{q},-\omega,\vec{B}) = \sigma_{ba}(\vec{q},-\omega,-\vec{B})$, or $(\vec{q}\to-\vec{q},\ \omega\to-\omega)$

$$\boxed{\sigma_{ab}(\vec{q},\omega,\vec{B}) = \sigma_{ba}(-\vec{q},\omega,-\vec{B})}$$ – Onsager relations for conductivity.

Consequences:

a) $\vec{q}=0,\ \vec{B}=0$: $\sigma_{ab}(\omega) = \sigma_{ba}(\omega)$

b) $\vec{q}=0,\ \vec{B}\neq 0$: $\sigma_{ab}(\omega,\vec{B}) = \sigma^{\text{Drude}}_{ab}(\omega) + \chi_{abc} B_c$,

$$\boxed{\chi_{abc} = -\chi_{bac}}$$ – Hall effect.

c) $\vec{q}\neq 0,\ \vec{B}=0$: $\sigma_{ab}(\omega,\vec{q}) \simeq \sigma^{\text{Drude}}_{ab}(\omega) + \lambda_{abc} q_c$,

$$\boxed{\lambda_{abc} = -\lambda_{bac}}$$ – Natural optical activity.

d) $\vec{q} \neq 0$, $\vec{B} \neq 0$: $\sigma_{ab}(\omega, \vec{q}, \vec{B}) \simeq \sigma_{ab}^{Drude}(\omega) + g_{abcd}\, q_c B_d$,

$$\boxed{g_{abcd} = g_{bacd}} \text{ - gyrotropic birefringence.}$$

Example: "dynamic chiral magnetic effect", or "natural optical activity in metals".

We have seen that CME is impossible in equilibrium, but it turns out one can have $\vec{j} = \lambda \vec{B}(t)$ in response to an oscillating field: "dynamic CME". [Response to oscill. $\vec{B}$-field!!! Not static!]

At a finite frequency, we have

$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$ (Faraday's law), hence $\vec{B}_\omega = \frac{\vec{q} \times \vec{E}}{\omega}$. This implies that

$$\vec{j}_{dCME} = \lambda \frac{\vec{q} \times \vec{E}}{\omega}:$$ "d CME" is nothing but optical activity!

$j_{a,dCME} = \lambda/\omega \cdot \varepsilon_{abc} q_c E_b$, or $\sigma_{dCME} = \frac{\lambda}{\omega} \varepsilon_{abc} q_c$ - as expected.

In a non-isotropic case, we have (drop "dCME" subscript)

$$j_a = \Lambda_{abc}\, q_c E_b, \quad \underline{\Lambda_{abc} = -\Lambda_{bac}} \text{ by Onsager relations.}$$

Where does Λ come from? It turns out to be related to the quasiparticle orbital moment. Part of the effect comes from velocity renormalization:

$$E_{np} = \varepsilon_{np} - \vec{\mu}_{np} \cdot \vec{B}, \text{ hence } \vec{v}^{\,tot}_{np} = \underbrace{\partial_{\vec{p}}\, \varepsilon_{np}}_{=\vec{v}_{np}} - \partial_{\vec{p}} (\vec{\mu}_{np} \cdot \vec{B})$$

If $\omega \gg \frac{1}{\tau}$, we can neglect relaxation, and assume that energy levels are populated according to old d.f. with old energy, ε_{np}, which existed before we turned on the oscillating field. Then the current is

$$\vec{j} = e\int_p f_{eq}(\varepsilon_{np}) \cdot (\cancel{\partial_{\vec{p}}\, \varepsilon_{np}} - \partial_{\vec{p}}\, \vec{\mu}_{np} \cdot \vec{B}) \overset{\text{by parts}}{=} e\int_p \left[\vec{\mu}_{np} \cdot \vec{B}\right] \partial_{\vec{p}} f_{eq}(\varepsilon_{np})$$

gives ϕ

crucial that it is ε_{np}, not E_{np} entering here, which is the case at high ω.

$$\vec{j}_a = -\int_p e v_a \cdot \mu_d \cdot B_d \left[-\frac{\partial f_{eq}}{\partial \varepsilon} \right] = -\frac{1}{\omega} \int_p v_a \mu_d \varepsilon_{dcb} q_c E_b \left[-\frac{\partial f_{eq}}{\partial \varepsilon} \right]$$

$$\sigma_{ab} = \frac{e}{\omega} \cdot \int_p v_a \mu_d \varepsilon_{dbc} q_c \left[-\frac{\partial f_{eq}}{\partial \varepsilon} \right]$$ – this is not anti-sym. w.r.t. $a \leftrightarrow b$!

Onsager relations tell us immediately that we are missing part of an effect!

The other part of the current appears as the magnetization current: $i\vec{q}$ in Fourier space

$\vec{j}_M = \vec{\nabla} \times \vec{M}$, where $\vec{M}$ is the magnetization induced by the electric field associated with the time-changing $\vec{B}$-field.

$$\vec{M} = \int_p \vec{\mu}_{np} \cdot \delta f_p , \quad \delta f_p = \frac{1}{i\omega} \cdot e\vec{E} \cdot \partial_{\vec{p}} f_{eq} = \frac{1}{i\omega} \cdot e\vec{E}\, \vec{v}_{np} \frac{\partial f_{eq}}{\partial \varepsilon_{np}}$$

$$j_{M,a} = \frac{e}{\omega} \cdot q_c \varepsilon_{acd} \int_p \cdot \mu_d \cdot v_b \frac{\partial f_{eq}}{\partial \varepsilon_{np}} E_b .$$

Combined with the previously considered part of the current, the conductivity tensor becomes

$$\sigma_{ab}^{dCME} = \frac{e}{\omega} \cdot \int_p \left[v_a \mu_d \varepsilon_{dbc} q_c - v_b \mu_d \varepsilon_{dac} q_c \right] \left[- \frac{\partial f_{eq}}{\partial \varepsilon_{\mathbf{p}}} \right] -$$

perfectly antisymmetric, as needed.

<u>Lesson:</u> always keep in mind the Onsager relations, and magnetization currents!