Semiclassical transport
in
metals with band geometry

Dima Pesin
University of Virginia, Charlottesville, USA

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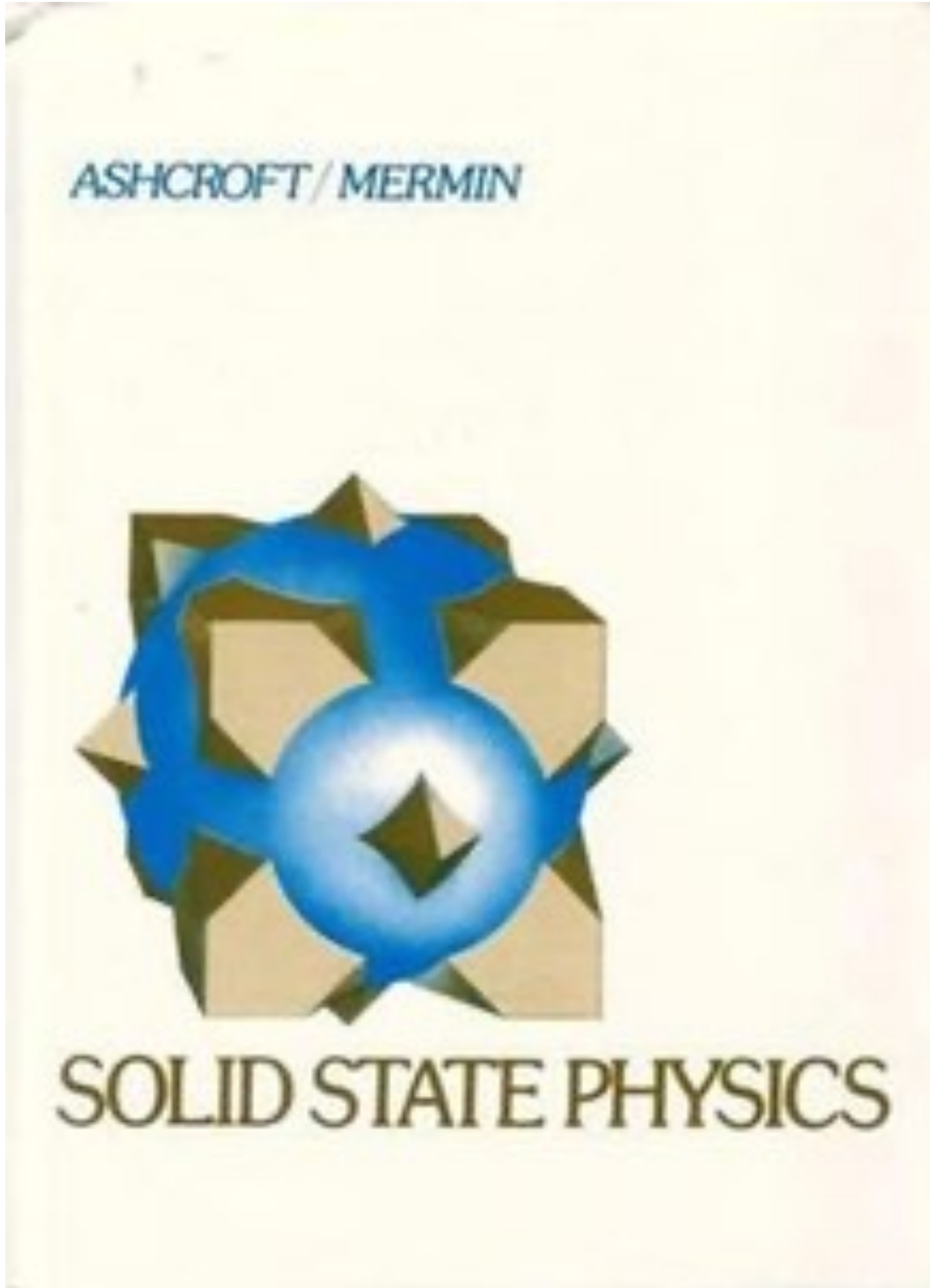
Lecture 1: Semiclassical motion and quantum osc.

- Semiclassical motion of band electrons
- Semiclassical (Lifshitz-Onsager) quantization
- Quantum magneto-oscillation phenomena

Lecture 2: The best five hours of your life

- Boltzmann equation. Electron (impurity) scattering
- Magnetoresistance in metals
- Chiral anomaly and chiral magnetic effect in WSMs
- Anomaly-induced negative LMR
- Static and dynamic CME. Onsager relations.
- Extrinsic contributions to the AHE: side jump, skew scattering.

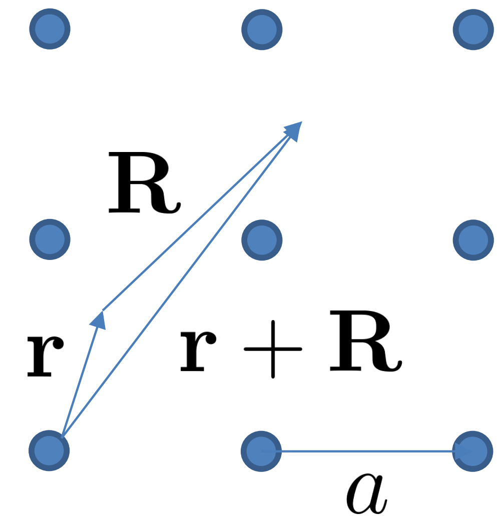
Lecture notes for Lecture 1:



Band theory of solids

Lattice translation symmetry:

$$[H, T(\mathbf{R})] = 0, \quad T(\mathbf{R})|\Psi\rangle = e^{i\mathbf{k}\mathbf{R}}|\Psi\rangle$$



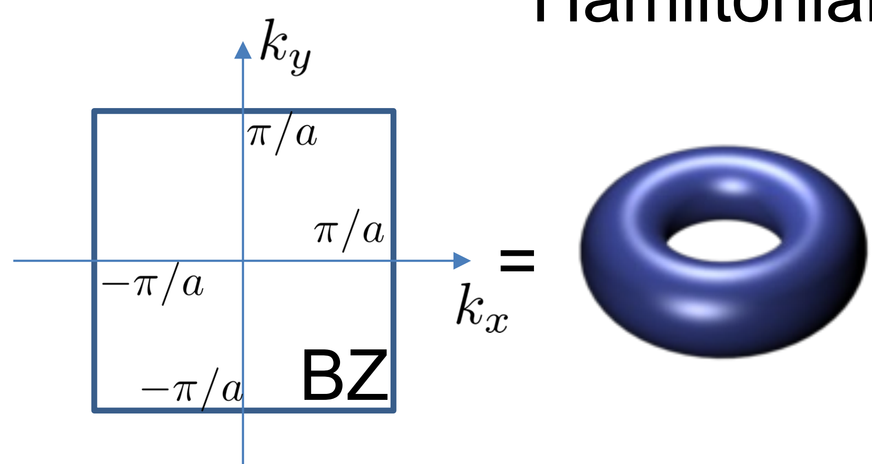
Bloch theorem: $\Psi_{n\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\mathbf{r}} u_{n\mathbf{k}}(\mathbf{r})$

Schrödinger equation:

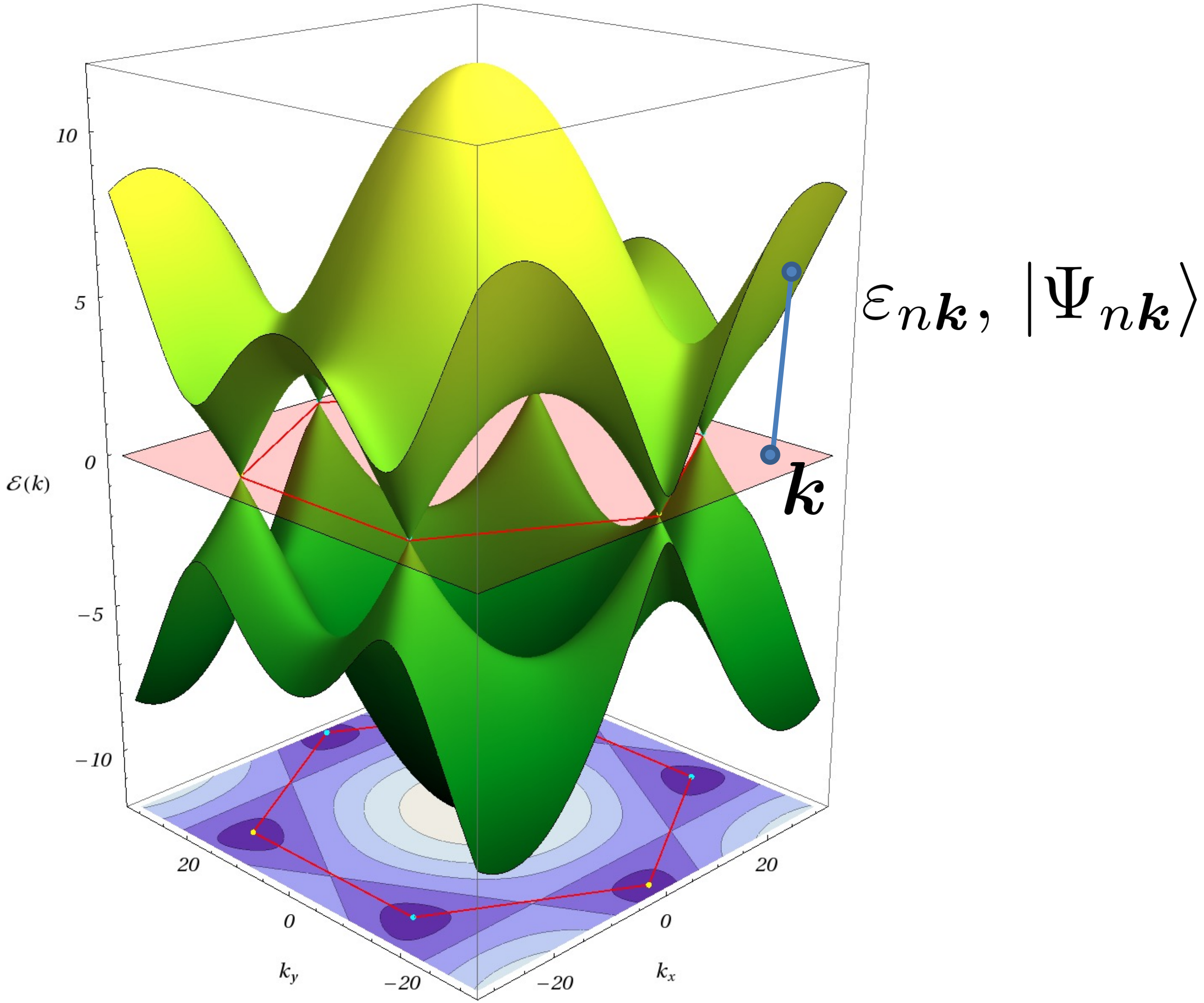
$$H|\Psi\rangle = E_{n\mathbf{k}}|\Psi\rangle$$

$$H_{\mathbf{k}}|u_{n\mathbf{k}}\rangle = E_{n\mathbf{k}}|u_{n\mathbf{k}}\rangle, \quad H_{\mathbf{k}} = e^{-i\mathbf{k}\mathbf{r}} H e^{i\mathbf{k}\mathbf{r}} -$$ Bloch Hamiltonian

$\mathbf{k}$ - “quasimomentum”.
Physically distinct ones
belong to the Brillouin zone:

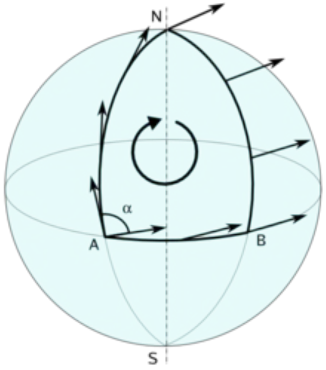


Band theory yields band structure



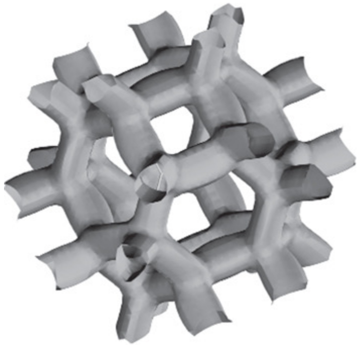
Two types of geometry in metals

a) "Lifshitz-Azbel-Kaganov" geometry:
geometry of iso-energetic surfaces,
$E_{n\mathbf{k}} = \mathrm{const}$, led to "Fermiology"



$$d\alpha = A_\theta d\theta + A_\varphi d\varphi$$

\+



Fermi surface of Pb

b) "Pancharatnam-Berry" geometry:
geometry of wave functions

$$\gamma_C = \int_C d\mathbf{k} \cdot \mathbf{A_k}$$

$$\mathbf{A_k} = i\langle u_\mathbf{k} | \nabla_\mathbf{k} | u_\mathbf{k} \rangle$$ - Berry connection

$e^{i\gamma_C} |u_{\mathbf{k}_f}\rangle$

C

$|u_{\mathbf{k}_i}\rangle$

Classical mechanics of electrons in solids

Promote the band energy to the classical Hamiltonian:

$$H(\boldsymbol{r}, \boldsymbol{k}) = E_{\boldsymbol{k}-e\boldsymbol{A}} + e\phi(\boldsymbol{r})$$

"Peierls substitution"

Write down the equations of motion:

$$\dot{\boldsymbol{r}} = \partial_{\boldsymbol{k}} E_{\boldsymbol{k}},$$

$$\dot{\boldsymbol{k}} = -\partial_{\boldsymbol{r}} E_{\boldsymbol{k}} + e\partial_{\boldsymbol{k}} E_{\boldsymbol{k}} \times \boldsymbol{B}.$$

Then perhaps solve the Boltzmann equation:

$$\partial_t f + \dot{\boldsymbol{r}} \boldsymbol{\nabla} f + \dot{\boldsymbol{k}} \partial_{\boldsymbol{k}} f = \hat{I}_{st}$$

Lecture 2 will discuss the BE, also deviations from this picture, the departure from the classical point of view.

Semiclassical trajectories

$\vec{E} \neq 0,\ \vec{B} = 0$

In this case

$$\left.\begin{aligned} \dot{\vec{r}} &= \frac{\partial \varepsilon_p}{\partial \vec{p}}, \\ \dot{\vec{p}} &= e\vec{E} \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{p} &= \vec{p}_0 + e\vec{E}t \\ \vec{r} &= \vec{r}_0 + \int^{t} dt' \, \vec{v}_{\vec{p}(t')} \end{aligned}$$

The equations look like motion in free space, but the actual motion is very far from that. Example is the 1D Bloch oscillations:

$\vec{p} \to p$, $\vec{r} \to x$, $\varepsilon_p = \varepsilon_0 - 2\Delta \cos\frac{pa}{\hbar}$ → 1D tight-binding dispersion.

We have

$p = \cancel{p_0 +}\, eEt$: uniform motion around the circular BZ.

$$\frac{dx}{dt} = \frac{2a\Delta}{\hbar} \cdot \sin\frac{eEta}{\hbar} \Rightarrow x = x_0 - \frac{2\Delta}{eE}\cos\frac{eEta}{\hbar} \qquad \left(x(0) = x_0 - \frac{2\Delta}{eE} \right)$$

Motion in real space is oscillatory with frequency $\frac{eEa}{\hbar}$. These are never observed because scattering destroys the oscillations before a single period can be completed ($\frac{\hbar}{eEa} \sim \tau \Rightarrow E \sim \frac{10^{-34}}{10^{-19}} \cdot \frac{1}{10^{-12}} \cdot \frac{1}{10^{-10}} \sim 10^{7}\ \mathrm{V/m}$.)

$\vec{E}=0,\ \vec{B}\neq 0$ $\quad \vec{B}=(0,0,B)$

In this case,

$$\dot{\vec{p}} = e\vec{v}_{\vec{p}}\times\vec{B} \Rightarrow \dot{p}_z = 0,\ p_z = \text{const}$$

$$\dot{\varepsilon}_p = \frac{\partial \varepsilon_p}{\partial p}\cdot\dot{\vec{p}} = \vec{v}_p\cdot\dot{\vec{p}} = 0 \Rightarrow \varepsilon_p = \text{const}.$$

motion is along cross sections of isoenergetic surfaces by a plane $p_z = \text{const}$ (which is $\perp \vec{B}$)

If semiclassical orbits are closed, we expect energy quantization in the quantum case: Landau levels.

We can obtain the corresponding quantization condition from a "Bohr"-type treatment, known as Lifshitz-Onsager quantization in the context of band theory.

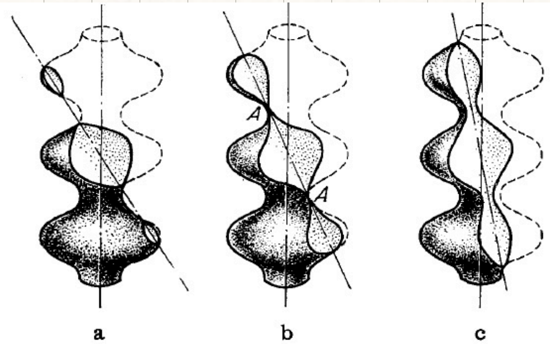
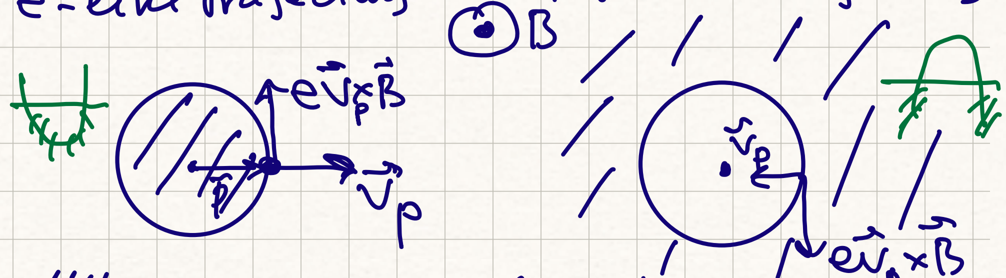


FIG. 8. Change in the character of the electron trajectory as a function of the angle between the magnetic field and the axis of the crimped cylinder. The points A are the saddle points.

Direction of motion: $e<0$!

e-like trajectory $\qquad$ h-like trajectory



//// – energies smaller than ε

Lifshitz - Onsager quantization

$H = \mathcal{E}(\vec{p} - e\vec{A})$, $\vec{A} = (-By, 0, 0)$, $\vec{B} = \vec{\nabla}\times\vec{A} = (0,0,B)$

Classical trajectory in momentum space: $\mathcal{E}(\vec{p}_\perp, p_z) = \text{const}$

In real space the trajectory is found from $d\vec{p}_\perp = e\, d\vec{r}\times\vec{B}$ (at least its projection onto the plane $\perp \vec{B}$)

Quantization condition:

$\frac{1}{\hbar}\oint \vec{p}_\perp \cdot d\vec{r} + \varphi_{Berry} = 2\pi(n+\gamma)$, γ depends on the spectrum, and comes from the turning point. $\gamma = \frac{1}{2}$ for quadratic dispersion.

↳ see D. Vanderbilt's lecture

To evaluate $\oint \vec{p}\, d\vec{r}$, note that $\vec{p}_\perp \cdot d\vec{r} = \vec{p}_\perp \cdot d\vec{r}_\perp$, and

$d\vec{p}_\perp = e\, d\vec{r}_\perp \times \vec{B}$, hence $\vec{B}\times d\vec{p}_\perp = e\,\vec{B}\times(d\vec{r}_\perp\times\vec{B}) = e[B^2 d\vec{r}_\perp - \vec{B}(\cancel{\vec{B}\cdot d\vec{r}_\perp})]$, or

$d\vec{r}_\perp = \frac{1}{eB^2}\cdot\vec{B}\times d\vec{p}_\perp$, and $\vec{p}_\perp\cdot d\vec{r}_\perp = \frac{1}{eB}\cdot\vec{p}_\perp\cdot\vec{e}_B\times d\vec{p}_\perp = \frac{-1}{eB}\cdot\vec{e}_B\cdot\vec{p}_\perp\times d\vec{p}_\perp$.

But $\vec{p}_\perp\times d\vec{p}_\perp = \pm d\vec{S}$, $d\vec{S}$ being element of the surface swept by the trajectory, $\pm$ depends on the sense of going around the trajectory.

Let us assume electron-like trajectory, for which $\vec{p}_\perp \times d\vec{p}_\perp$ is parallel to $\vec{B}$. Then we obtain

$$\oint \vec{p}_\perp \cdot d\vec{r}_\perp = \frac{1}{|e|B} \cdot S(\mathcal{E}, p_z)$$, $S(\mathcal{E}, p_z)$: area swept by the trajectory in the momentum space.

We need to remember that the conserved energy is

$E_p = \mathcal{E}_p - \vec{\mu}_p \vec{B}$, where $\mathcal{E}_p$ is the band energy. It is E_p that gets quantized. Let us neglect the Berry phase and the magnetic moment to obtain the standard Lifshitz-Onsager result:

$$\frac{1}{|e|B} S(\mathcal{E}, p_z) = 2\pi(n+\gamma)\hbar, \text{ or}$$

$$\boxed{S(\mathcal{E}, p_z) = 2\pi\hbar|e|B(n+\gamma)}$$ - Lifshitz-Onsager quantization condition.

Level spacing: $\hbar\omega_n = \mathcal{E}_n - \mathcal{E}_{n-1}$:

$$S(\mathcal{E}_n, p_z) - S(\mathcal{E}_{n-1}, p_z) \simeq \frac{\partial S}{\partial \mathcal{E}} \cdot \hbar\omega_n = 2\pi\hbar|e| \cdot B, \text{ or } \hbar\omega_n = \hbar\frac{|e|B}{m_c}, \text{ where}$$

$$\boxed{m_c = \frac{1}{2\pi} \cdot \frac{\partial S}{\partial \mathcal{E}}}$$ - cyclotron mass.

Check: $\varepsilon = \frac{p^2}{2m}$, $S = \pi p_\perp^2 = \pi\cdot(2m\varepsilon - p_z^2)$, $\gamma = \frac{1}{2}$. Then

$$\varepsilon_n = \frac{\hbar|e|B}{m}\left(n+\tfrac{1}{2}\right) + \frac{p_z^2}{2m}$$ – the standard LL; $m_c = \frac{1}{2\pi}\frac{\partial S}{\partial \varepsilon} = m$

check: graphene, $\varepsilon = \pm v p$, $\varphi_{Berry} = \pi$, $\vec{\mu}_p = 0$:

$$S(\varepsilon) = \pi p^2 = \frac{\pi \varepsilon^2}{v^2}$$

$$\frac{1}{\hbar|e|B}\cdot\frac{\pi\varepsilon^2}{v^2} + \cancel{\pi} = 2\pi\left(n + \cancel{\tfrac{1}{2}}\right) \Rightarrow \varepsilon = \pm\sqrt{2\hbar|e|v^2B\cdot n}$$ – the usual LL spectrum in graphene.

$$m_c = \frac{1}{2\pi}\frac{\partial S}{\partial \varepsilon} = \frac{\varepsilon}{v^2}$$ – cyclotron mass is given by energy

Check: Weyl metal, or "3D graphene": $H = v\vec{\sigma}\vec{p}$, $\varepsilon = \pm\sqrt{v^2 p_\perp^2 + v^2 p_z^2}$

Easy way: $H_{DM} = v\vec{\sigma}(\vec{p} - e\vec{A}) = \underbrace{v\vec{\sigma}_\perp(\vec{p}_\perp - e\vec{A})}_{\text{"graphene"}} + \underbrace{v\, p_z \sigma_z}_{\text{"gap"}} \rightarrow \varepsilon_{n\neq 0} = \pm\sqrt{2\hbar|e|v^2Bn + v^2p_z^2}$

$\varepsilon_{n=0} = p_z v$

(0th LL is chiral!)

Our way: $\varepsilon = v\sqrt{p_\perp^2 + p_z^2}$

$$S(\varepsilon, p_z) = \pi\cdot p_\perp^2 = \pi\left(\frac{\varepsilon^2}{v^2} - p_z^2\right);\quad \varphi_B = \int d\vec{S}_\perp\cdot\vec{\Omega}_p = -\pi\,\mathrm{sgn}\,p_z\left(1 - \frac{v|p_z|}{\varepsilon}\right)$$

What is going on?

Hint:

Pauli: spin is not self-rotation!

The physical interpretation of Pauli's "degree of freedom" was initially unknown. Ralph Kronig, one of Landé's assistants, suggested in early 1925 that it was produced by the self-rotation of the electron. When Pauli heard about the idea, he criticized it severely, noting that the electron's hypothetical surface would have to be moving faster than the speed of light in order for it to rotate quickly enough to produce the necessary angular momentum. This would violate the theory of relativity. Largely due to Pauli's criticism, Kronig decided not to publish his idea.

LL vol 3, "The current density in a magnetic field":

Comparing this expression with (115.1), we find the following expression for the current density:

$$\mathbf{j} = \frac{ie\hbar}{2m}[(\nabla\Psi^*)\Psi - \Psi^*\nabla\Psi] - \frac{e^2}{mc}\mathbf{A}\Psi^*\Psi + (\mu/s)c\,\mathbf{curl}(\Psi^*\hat{\mathbf{s}}\Psi). \quad (115.4)$$

If spin is not self-rotation, why does spin magnetization contribute to the current?!

Hint:

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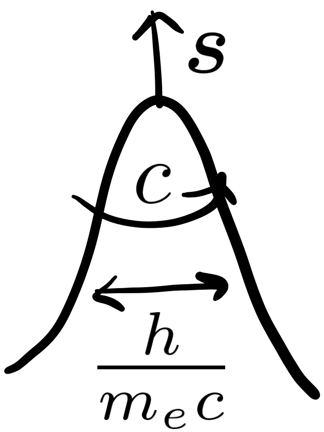
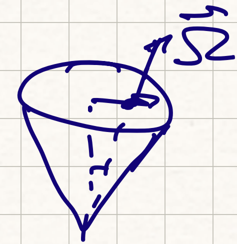
Resolution:

$$\boldsymbol{m} = g\frac{e\hbar}{2m_e}\boldsymbol{s} \leftrightarrow$$



See also: effective g-factor in semiconductors

LL in Weyl metals



$$E = vp - \vec{\mu}_B \vec{B}.$$

$$\vec{\Omega} = -\frac{1}{2}\frac{1}{p^2}\vec{e}_p\,; \quad \varphi_B = \int d\vec{S}\cdot\vec{\Omega} = -\frac{1}{2}\int d\varphi \cdot p_\perp dp_\perp \cdot \frac{1}{p^2}\cdot \cancel{\cos\theta}\ \frac{p_z}{p}$$

$$= -\frac{1}{2}\int d\varphi \cdot dp_\perp \cdot p_\perp \frac{p_z}{p^3} = -\pi \cdot \int_0^{p_0^2-p_z^2} dp_\perp\, p_\perp \frac{p_z}{(p_\perp^2+p_z^2)^{3/2}}$$

$$= -\pi\,\mathrm{sgn}\,p_z \int_0^{\frac{p_0^2}{p_z^2}-1} dx\cdot x\cdot \frac{1}{(1+x^2)^{3/2}} \begin{cases} \overset{p_z\to 0}{=} -\pi\,\mathrm{sgn}\,p_z \cdot \int_0^{\infty} dx\cdot x\cdot \frac{1}{(1+x^2)^{3/2}} \\ = -\pi\,\mathrm{sgn}\,p_z \cdot \checkmark \end{cases}$$

$$\varphi_B = -\pi\,\mathrm{sgn}\,p_z \cdot \left(1 - \frac{1}{\sqrt{\frac{p_0^2}{p_z^2}-1+1}}\right) = -\pi\,\mathrm{sgn}\,p_z\left(1-\frac{|p_z|}{p_0}\right)$$

$$E = \varepsilon_p - \vec{\mu}\vec{B} \Rightarrow \varepsilon_p = E + \vec{\mu}\vec{B}$$

What is $\vec{\mu}_p$?

Long way: $\vec{\mu}_p = \frac{ie}{2}\langle \partial_{\vec{p}} u_{np}| \times (H_{np} - \varepsilon_{np}) | \partial_{\vec{p}} u_{np}\rangle$

conduction band

Short way for a Weyl fermion: $\vec{L}_p = \cancelto{0}{\langle \vec{r}\times\vec{p}\rangle} + \frac{\hbar}{2}\langle\vec{\sigma}\rangle = \frac{\hbar}{2}\vec{e}_p$

$$\vec{\mu}_p = \frac{e}{\text{"}m\text{"}}\cdot\vec{L}_p = \frac{e}{\varepsilon/v^2}\cdot\frac{\hbar}{2}\vec{e}_p = \frac{\hbar}{2}\frac{ev}{p}\vec{e}_p$$

Semiclassical quantization:

$$\frac{1}{|e|B}S(E+\vec{\mu}\vec{B}) + \varphi_B = 2\pi\hbar\left(n+\frac{1}{2}\right), \quad S(\varepsilon) = \pi\left(\frac{\varepsilon^2}{v^2} - p_z^2\right), \quad \varphi_B = -\pi\,\mathrm{sgn}\,p_z\left(1-\frac{|p_z|}{p}\right)$$

$$\frac{\partial S}{\partial\varepsilon}\cdot\vec{\mu}\vec{B} = 2\pi\cdot\frac{\varepsilon}{v^2}\cdot\frac{ev}{2p}\cdot\frac{p_z}{p}\cdot B = \pi\frac{eB}{2p}\cdot p_z, \text{ so } \frac{1}{|e|B}\cdot\frac{\partial S}{\partial\varepsilon}\vec{\mu}\vec{B} + \varphi_B = -\pi\,\mathrm{sgn}\,p_z,$$

and one gets back to the usual result.

Magneto-oscillation phenomena

First, we consider the de Haas - van Alphen effect (1930's) and focus on a 2DEG with quadratic dispersion: $\varepsilon_p = \frac{p^2}{2m}$.

As is well known, in the presence of a B-field $\perp$ to the plane of the sample, the energy spectrum is

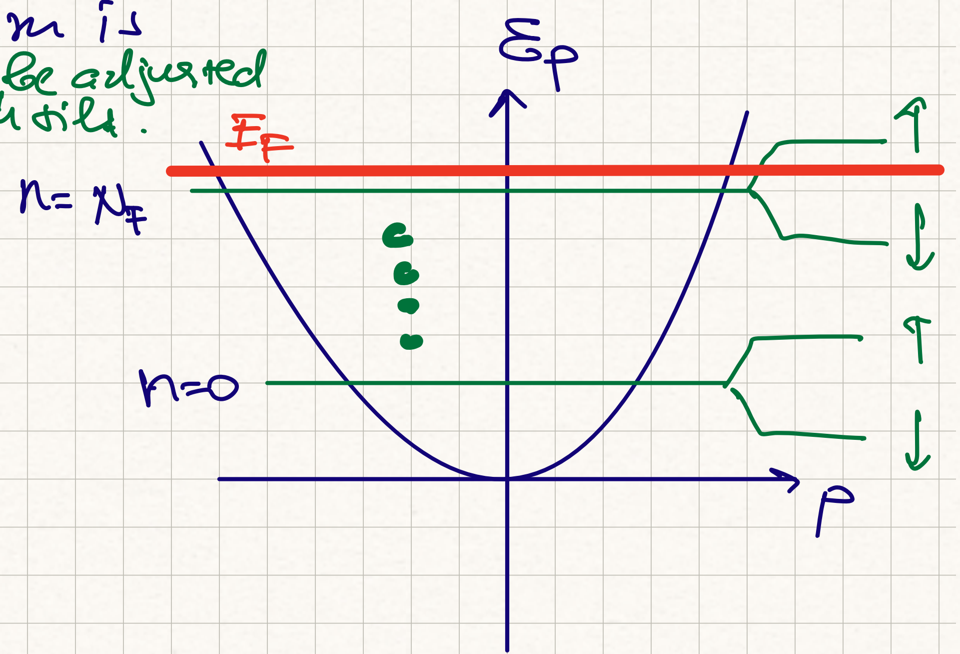
$$\varepsilon_{n\sigma} = \hbar\omega_c\left(n+\tfrac{1}{2}\right) + \tfrac{1}{2} g \mu_B \sigma \cdot B \,,$$

can be adjusted with tilt.

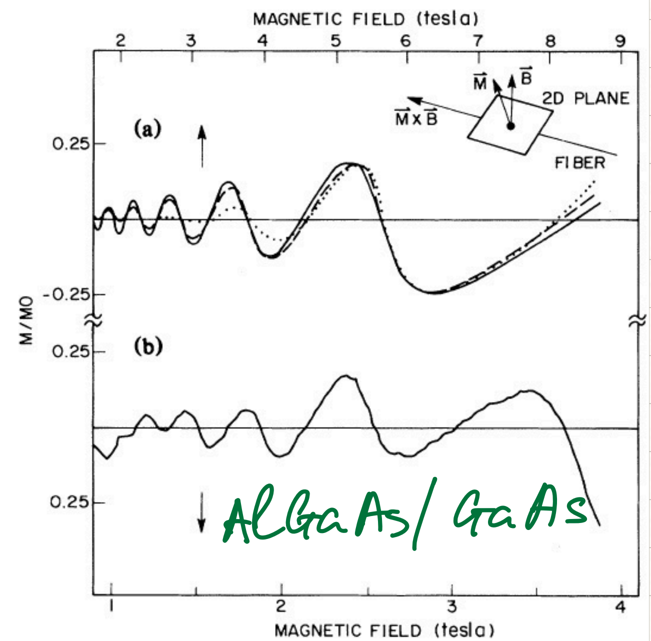
where n labels the Landau levels, σ is the spin index, and

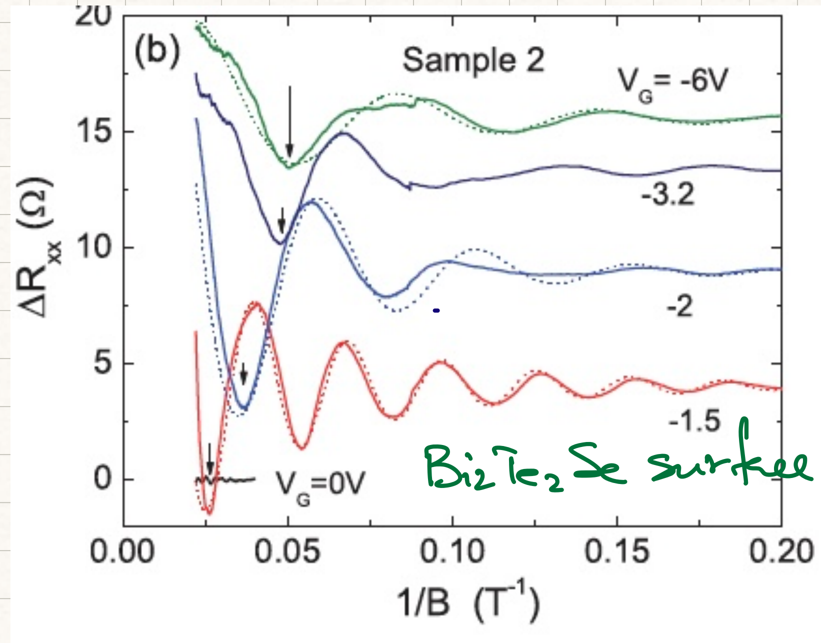
$$\mu_B = \frac{e\hbar}{2m_e}, \quad \omega_c = \frac{eB}{m}$$

not the same!



It turns out that various thermodynamic and transport quantities oscillate as a function of B-field (or rather $1/B$)





J.P. Eisenstein et al., PRL 1985

Xiong et. al., PRB 2013

The origin of the oscillations can be deduced from keeping track of LL occupation as B is increased. It is clear that for B such that E_F is in the gap betwee LLs, their occupations are smooth functions of B-field: each LL has $\frac{\Phi}{\Phi_0}$ electrons in it (Φ – flux through the sample) (Φ_0 → flux quantum), when a LL crosses the Fermi level, it rapidly loses all of its electrons, causing a sudden change in the systems energy. This leads to oscillations in quantities like $\vec{M}$, or χ-susceptibility -, which are derivatives of the thermodynamic potential w.r.t. B-field.

Oscillations of transport quantities - Shubnikov-de Haas effect - can be understood in a similar way, by noting the motion of LL through the Fermi level lead to oscillations of the DoS at E_F. Since it determines scattering rates by the Fermi Golden Rule, we expect oscillations in the transport coefficients.

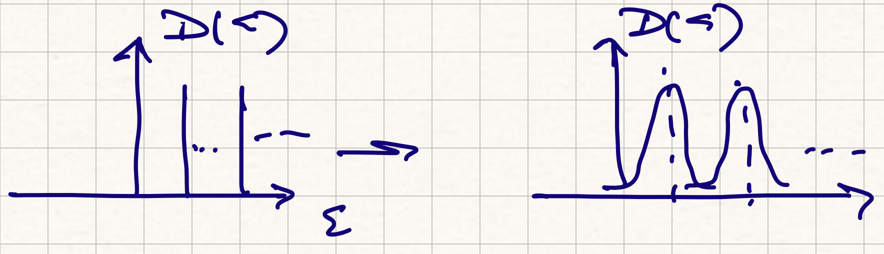
Oscillations of the DoS

Thermodynamic properties require knowledge of the DoS(ε) only, since for fermions

$$\Omega = -T \int d\varepsilon \underbrace{\mathcal{D}(\varepsilon)}_{\text{DoS}} \ln\left(1 + e^{\frac{\mu-\varepsilon}{T}}\right).$$

DoS oscillations can also explain qualitatively the osc. in transport quantities.

Let us assume that each LL has a Lorentzian shape:



with some width Γ (can be level-dependent, but not this time)
as well as B-dependent

$$D(\varepsilon, B) = \sum_{n\sigma} \frac{\Phi}{\Phi_0} \delta(\varepsilon - \varepsilon_{n\sigma}) \to \sum_n \Phi/\Phi_0 \, D_L(\varepsilon - \varepsilon_{n\sigma}), \quad (*)$$

$$D_L = \frac{1}{\pi} \frac{\Gamma/2}{(\varepsilon - \varepsilon_{n\sigma})^2 + \frac{1}{4}\Gamma^2}.$$

To keep things clear, let us change $\hbar\omega_c(n+1/2) \to \hbar\omega_c(n+\gamma)$, so we can see what role is played by the offset of the LL index.

The problem with sum (*) is that many n contribute to it, and that it consequently contains both the osc. parts of the DoS coming from the vicinity of the FS, as well as the smooth part of it, coming from the entire Fermi sea. We need to switch to some "dual space" to make progress.

Poisson summation formula

Consider the periodic δ-function:

$\tilde{\delta}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$. Clearly $\tilde{\delta}(x+m) = \tilde{\delta}(x)$, $m \in \mathbb{Z}$. So it is a periodic function with period 1. We can thus expand it in a Fourier series:

$$\tilde{\delta}(x) = \sum_{k=-\infty}^{\infty} e^{-2\pi i k x} \,:\quad \int_{-\frac{1}{2}}^{1/2} dx\, \tilde{\delta}(x)\, e^{+ik'2\pi x} = \int_{-1/2}^{1/2} dx\, \delta(x)\, e^{+ik'2\pi x} = 1. \checkmark$$

Then we obtain the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} F(n+y) = \int_{-\infty}^{\infty} dx\, \tilde{\delta}(x) \cdot F(x+y) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} F(x+y)$$

assumed that the integral and the sum can be interchanged.

$$= \sum_{k=-\infty}^{\infty} e^{+2\pi i k y} \cdot \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} F(x) \equiv \sum_{k=-\infty}^{\infty} e^{2\pi i k y} \cdot F_k, \quad F_k = \int dx\, e^{-2\pi i k x} f(x).$$

Separating $k=0$ from the rest of harmonics, we get

$$\boxed{\sum_{n=-\infty}^{\infty} F(n+\gamma) = \underbrace{\int_{-\infty}^{\infty} dx\, F(x)}_{\text{smooth part.}} + \sum_{k=1}^{\infty} e^{2\pi i k\gamma} F_k + \text{c.c.}}$$

In our case, $F(x+\gamma) = \mathcal{D}(\varepsilon - \varepsilon_{x\sigma}) \cdot \underbrace{\frac{1+\Theta(\varepsilon)}{2}}_{\text{to remove spurious negative energies introduced by } \sum_{k=-\infty}^{\infty}}$

Let us see what we obtain for the smooth part of the DoS:

plays no role for $\varepsilon \sim E_F$

$$\overline{\mathcal{D}_\sigma(\varepsilon)} = \int_{-\infty}^{\infty} dx \cdot \cancel{\frac{1+\Theta(\varepsilon)}{2}} \frac{1}{\pi} \frac{\Gamma/2}{(\varepsilon_\sigma - \hbar\omega_c x)^2 + \Gamma^2/4} \cdot \frac{\Phi}{\Phi_0} \quad \Big\} \ \varepsilon_\sigma = \varepsilon - g\mu_B B\sigma$$

$$= \frac{1}{\hbar\omega_c} \cdot \frac{\Phi}{\Phi_0} = \frac{1}{\hbar \cdot \frac{eB}{m}} \cdot \frac{B \cdot A}{h/e} = \underbrace{\frac{m}{2\pi\hbar^2}}_{\text{DoS per unit area.}} \cdot A \rightarrow \text{the usual 2D DOS for } \varepsilon_p = p^2/2m.$$

Now the oscillating part:

$$D_{k\sigma} = \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} \cdot \frac{1}{\pi} \cdot \frac{\Gamma/2}{(\varepsilon_\sigma - \hbar\omega_c x)^2 + \Gamma^2/4} \cdot \frac{\Phi}{\Phi_0} = \quad \Big\} \; \varepsilon_\sigma = \varepsilon - g\mu_B B \cdot \sigma$$

$$= \underbrace{\frac{1}{\hbar\omega_c} \cdot \frac{\Phi}{\Phi_0}}_{\bar{D}} \cdot \int_{-\infty}^{\infty} dx\, e^{-2\pi i k \cdot \frac{x}{\hbar\omega_c}} \cdot \frac{1}{\pi} \cdot \frac{\Gamma/2}{(\varepsilon_\sigma - x)^2 + \Gamma^2/4} = \text{(use residues)}$$

$$= \bar{D} \cdot \frac{1}{\pi} (-2\pi i) \cdot e^{-\pi k \frac{\Gamma}{\hbar\omega_c}} \cdot e^{-2\pi i k \frac{\varepsilon_\sigma}{\hbar\omega_c}} \cdot \frac{\Gamma/2}{2(-i\Gamma/2)}$$

$$= \bar{D} \cdot e^{-\pi k \frac{\Gamma}{\hbar\omega_c}} \, e^{-2\pi i k \frac{\varepsilon_\sigma}{\hbar\omega_c}} \; ; \qquad \sum_\sigma e^{-2\pi i k \cdot \frac{-g\mu_B B \sigma}{\hbar\omega_c}}$$

The DoS is then $\sum_{k\sigma} D_{k\sigma}$:

disorder

spin/Zeeman splitting

LLs

$$D(\varepsilon, B) = 2\bar{D} + 4\bar{D} \cdot \sum_{k=1}^{\infty} e^{-\pi k \frac{\Gamma}{\hbar\omega_c}} \cdot \cos\left[2\pi k \cdot g \cdot \frac{m}{m_e}\right] \cdot \cos\left[2\pi k \left(\frac{\varepsilon}{\hbar\omega_c} - \gamma\right)\right]$$

↳ from spin

clearly, there are oscillations at given ε as a function of $\frac{1}{\hbar\omega_c} \propto \frac{1}{B}$.

The LL width, Γ, leads to the suppression of oscillations, and for $\Gamma \sim \hbar\omega_c$ they disappear altogether.

N.B. $\Gamma \equiv \frac{1}{\tau_q}$ is not the same as $\frac{1}{\tau_{tr}}$. Any scattering, not just large-momentum, contributes to the lifetime of a level, so in general $\tau_q < \tau_{tr}$ ("q" for "quantum").

Magnetization oscillations

Finally, we calculate $\vec{M} = \left(-\frac{\partial \Omega}{\partial \vec{B}}\right)_\mu \frac{1}{A}$ – magnetization for a fixed chemical potential. Note that we could also consider $\vec{M} = \left(-\frac{\partial F}{\partial \vec{B}}\right)_N \frac{1}{A}$ – fixed number of particles, which would include oscillations of μ. Which one to look at is determined by the experiment. If there is a reservoir to pin μ (like the bulk states for a 2D topological surface), then consider $\mu = \text{const}$, etc.

$$\Omega = \int' d\varepsilon\, D(\varepsilon, B)\left[-T \ln\left(1 + e^{\frac{\mu-\varepsilon}{T}}\right)\right] \frac{1+\Theta(\varepsilon)}{2}$$

We are interested in the oscillating part of Ω only, call it $\tilde{\Omega}$.
Using the expression for D, and integrating by parts twice, we obtain:

$$\int_{-\infty}^{\infty} d\varepsilon \cos\left(2\pi k \cdot \left(\frac{\varepsilon}{\hbar\omega_c} - \gamma\right)\right) \cdot \left[-T \ln(\ldots)\right] \frac{1+\Theta(\varepsilon)}{2}$$

does not oscillate, throw away

$$= -\frac{\hbar\omega_c}{(2\pi k)} \int_{-\infty}^{\infty} d\varepsilon \sin\left(2\pi k\left(\frac{\varepsilon}{\hbar\omega_c} - \gamma\right)\right)\left[f_{eq}(\varepsilon)\, \frac{1+\Theta(\varepsilon)}{2} - T\ln(\ldots) \cdot \frac{1}{2}\delta(\varepsilon)\right]$$

$$f_{eq}(\varepsilon) = \frac{1}{e^{\frac{\varepsilon-\mu}{T}} + 1}$$

$$= \left(\frac{\hbar\omega_c}{2\pi k}\right)^2 \cdot \int_{-\infty}^{\infty} d\varepsilon \cos\left(2\pi k\left(\frac{\varepsilon}{\hbar\omega_c} - \gamma\right)\right)\left[-\partial_\varepsilon f_{eq}\right] \frac{1+\Theta(\varepsilon)}{2}.$$

Since $\partial_\varepsilon f_{eq}$ is peaked around $E_F \tilde{=} \mu$ for $T \ll E_F$, we can neglect $\frac{1+\Theta(\varepsilon)}{2} \to 1$ at this point.

Now we can perform the ε-integral by using the expression for the Fourier transform of $\partial_\varepsilon f_{eq}$:

$$\int d\varepsilon \left(-\frac{\partial f_{eq}}{\partial \varepsilon}\right) e^{2\pi i k \frac{\varepsilon - E_F}{\hbar\omega_c}} = \frac{2\pi^2 k T/\hbar\omega_c}{\sinh(2\pi^2 k T/\hbar\omega_c)} \quad \text{(even in } k\text{)},$$

which brings $\tilde{\Omega}$ to the following form:

new suppression factor from T.

$$\tilde{\Omega} = \sum_{k=1}^{\infty} \cdot 4\bar{D} \cdot \left(\frac{\hbar\omega_c}{2\pi k}\right)^2 e^{-\frac{\pi\Gamma}{\hbar\omega_c}k} \cdot \cos\left(2\pi k g \frac{m}{m_e}\right) \cdot \frac{2\pi^2 k T/\hbar\omega_c}{\sinh(2\pi^2 k T/\hbar\omega_c)} \cdot \cos\left[2\pi k\left(\frac{E_F}{\hbar\omega_c} - \gamma\right)\right]$$

Finally, $\tilde{M}_z = -\frac{\partial \tilde{\Omega}}{\partial B_z} \cdot \frac{1}{A}$ (differentiate only the fast $\cos\left(2\pi k\left(\frac{E_F}{\hbar\omega_c} - \gamma\right)\right)$)
assume $\vec{B} = (0,0,B)$.

$$\tilde{M}_z \simeq + \frac{4\bar{D}}{A}\left(\frac{\hbar\omega_c}{2\pi}\right)^2 \cdot \sum_{k=1}^{\infty} \frac{2\pi \cancel{k}}{k^{\cancel{2}}} \cdot \frac{E_F}{\hbar\omega_c} \cdot \frac{1}{B} \; R_D \cdot R_Z \cdot R_T \cdot \sin\left(2\pi k\left(\frac{E_F}{\hbar\omega_c} - \gamma\right)\right)$$

$$= \frac{4\,\cancel{BA}}{\cancel{A}\,h/e} \cdot \frac{\cancel{1}}{\cancel{\hbar\omega_c}} \cdot \frac{\cancel{(\hbar\omega_c)^2}}{4\pi^2} \cdot \frac{2\pi E_F}{\cancel{\hbar\omega_c \cdot B}} \cdot \sum_{k=1}^{\infty} R_D \cdot R_Z\, R_T \, \frac{\sin\left(2\pi k\left(\frac{E_F}{\hbar\omega_c} - \gamma\right)\right)}{k}$$

$$\equiv \boxed{M_0 \cdot \sum_{k=1}^{\infty} e^{-\frac{\pi \Gamma}{\hbar\omega_c}\cdot k} \cos\left(2\pi k S \frac{m}{m_e}\right) \frac{2\pi^2 kT/\hbar\omega_c}{\sinh(2\pi^2 kT/\hbar\omega_c)} \cdot \frac{\sin\left(2\pi k\left(\frac{E_F}{\hbar\omega_c} - \gamma\right)\right)}{k}}$$

Lifshitz-Kosevich formula for a 2DEG.

$$[M_0] = \left[\frac{e \cdot E_F}{\hbar}\right] = \frac{[e]}{[T]} = \frac{[e]\cdot[L^2]}{T} \cdot \frac{1}{[L^2]} = \frac{(\text{magnetic moment})}{(\text{area})}$$

$$\hookrightarrow [e \cdot v_F \cdot \lambda_F] \cdot \left[\frac{1}{\lambda_F^2}\right]$$

Lessons:

1) LLs $\to$ oscillations of thermodyn. (M, $\chi = \frac{dM}{dB}$ ...), and transport ($\frac{\tilde{\sigma}}{\sigma_0} \sim \sqrt{\frac{\hbar\omega_c}{E_F}} \cdot \frac{\tilde{M}}{M_0}$) quantities as functions of $1/B$.

2) Oscillation frequency: $\frac{E_F}{\hbar\omega_c} = \frac{m E_F}{\hbar e \cdot B} = \frac{2\pi m \cdot E_F}{\Phi_0 \cdot B}$, determined by

$\pi \cdot 2m E_F \equiv \pi \cdot p_F^2$ – momentum space area inside the Fermi surface.

3) Oscillations are sensitive to disorder : $\Gamma \equiv \frac{1}{\tau_q}$.

temperature : T

spin splitting : g

LL energy offset : γ

FS area : $2m E_F \to S(E_F)$

All of these quantities can be extracted from experiment.

In 3D, FS area → extremal area of FS cross section ⊥ to $\vec{B}$.

There can be several such extremal (min or max) cross sections, and hence several frequencies. Allows reconstruction of the FS.

Generalization to 3D, arbitrary band structures

$$\boxed{M_0 \cdot \sum_{k=1}^{\infty} e^{-\frac{\pi T}{\hbar\omega_c}\cdot k} \cos\left(2\pi k S \frac{m}{m_e}\right) \frac{2\pi^2 kT/\hbar\omega_c}{\sinh(2\pi^2 kT/\hbar\omega_c)} \cdot \frac{\sin\left(2\pi k\left(\frac{E_F}{\hbar\omega_c} - \gamma\right)\right)}{k}}$$

L L_1 L_2

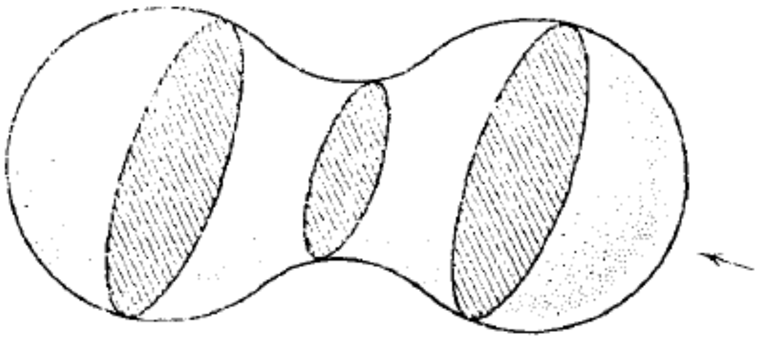


FIG. 12. Example of extremal cross sections of the Fermi surface. The direction of the magnetic field is indicated by the arrow.

Things needed:

$S(\varepsilon) \to S(\varepsilon, p_z)$,

$$\sum_{n\sigma} \to \sum_{n\sigma} \int \frac{dp_z}{2\pi\hbar}$$

$$m \to \frac{1}{2\pi} \cdot \frac{\partial S(\varepsilon, p_z)}{\partial \varepsilon}$$

$$\frac{2\pi E_F}{\hbar\omega_c} = \frac{2\pi m E_F}{\hbar |e| B} \to \frac{S(\varepsilon, p_z)}{\hbar |e| B}$$

$$\int \frac{dp_z}{2\pi} \left(\text{oscillations with varying phases}\right) \xrightarrow{\text{"stationary phase"!}} \sum_{\substack{\text{cross sections} \\ \text{with } \frac{\partial S}{\partial p_z} = 0}} \cdots \to \underline{\underline{\text{extremal}}}$$